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AN ORTHONORMAL BASIS FOR $C[0,1]$ THAT IS NOT AN UNCONDITIONAL BASIS FOR $L^p[0,1]$, $1 < p \neq 2$

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ABSTRACT. In a recent article, Kazaryan has employed an orthonormal system constructed by Olevskii in order to obtain a negative answer to the following question posed by Ulyanov: Is an orthonormal basis for $C[0,1]$ necessarily an unconditional basis for each space $L^p[0,1]$, $1 < p < \infty$? The elements of the Olevskii system, however, are finite linear combinations of Haar functions, and thus, most of them are not continuous on $[0,1]$. For this reason, the example is mildly unsatisfying, since one generally requires the members of a Schauder basis for a given space to belong to that space. In the present work, the author shows that this minute defect can be removed if one modifies the Olevskii system by replacing the Haar functions involved therein with corresponding members of the Franklin system.

1. A question posed by Ulyanov asks whether an orthonormal Schauder basis for $C[0,1]$ is necessarily an unconditional basis for each of the spaces $L^p[0,1]$, $1 < p < \infty$. Recently Kazaryan [**4**] has used a basis constructed by Olevskii [**5**] in order to answer this question in the negative. The elements of this basis are finite linear combinations of elements of the Haar system, and thus, for the most part, they are not elements of $C[0,1]$. In a picayune technical sense, such a system is not a Schauder basis for $C[0,1]$, since the standard definition of a basis for a Banach space requires the elements of the system to belong to the space. Thus, one is led to contemplate Ulyanov's question within the confines of this more restrictive framework. In the present note, the fundamental work of Ciesielski [**1**, **2**] is used to show that a system of Olevskii type in which the Franklin system plays the central role provides the answer to this refined version of the original question.

2. The Franklin system herein discussed is obtained by performing the Gram-Schmidt maneuver on the standard Schauder system, the family consisting of the constant function **1**, the identity function and the sawtooth functions whose supports are the dyadic intervals $[(k-1)2^{-m}, k2^{-m}]$, $m = 0, 1, \ldots; k = 1, \ldots, 2^m$. The individual Franklin functions are, thus, continuous and piecewise linear, and the Franklin system $\mathcal{F} = \{f_n : n = 0, 1, \ldots\}$ proves to be a Schauder basis for each space $L^p[0,1]$, $1 \leq p < \infty$, as well as for $C[0,1]$ (see, for example, [**1**]). Moreover, in his thorough analysis of the Franklin system, Ciesielski has established the following powerful inequalities.

THEOREM A (CIESIELSKI). *Let $N = 2^m$, with $m = 0, 1, \ldots,$ and let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence of real numbers. There is a positive constant A such that,*

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for each p in $[1,+\infty]$,

$$A^{-1}N^{(1/2-1/p)}\left(\sum_{n=N+1}^{2N}|a_n|^p\right)^{1/p} \le \left\|\sum_{n=N+1}^{2N} a_n f_n\right\|_p$$
$$\le AN^{(1/2-1/p)}\left(\sum_{n=N+1}^{2N}|a_n|^p\right)^{1/p}.$$

The Olevskii basis, to which reference has been made above, is constructed in blocks of sizes 2^m, $m=1,2,\ldots$, with the assistance of a sequence of orthogonal matrices defined by the following conditions. For each natural number m,

$$A_m=(a_{ij}^{(m)}), \qquad 1\le i,j\le 2^m,$$

where $a_{1j}^{(m)}=2^{-m/2}$ for $1\le j\le 2^m$; and, for $i=2^r+s$ with $0\le r\le m-1$ and $1\le s\le 2^r$,

$$a_{ij}^{(m)}=\begin{cases} 2^{-(m-r)/2}, & \text{if } (s-1)2^{m-r}+1\le j\le (2s-1)2^{m-r-1};\\ -2^{-(m-r)/2}, & \text{if } (2s-1)2^{m-r-1}+1\le j\le s2^{m-r};\\ 0, & \text{otherwise.}\end{cases}$$

The various blocks of the basis are obtained by applying the A_m to the canonical dyadic blocks of the Haar system. In the present work, one follows the same procedure using, however, the Franklin functions in place of the Haar functions, in order to obtain a continuous analogue of the Olevskii system.

For $n=0,1,2$, let $\varphi_n=f_n$; for $n=2^m+j$ with $m=1,2,\ldots$, and $1\le j\le 2^m$, let

$$\varphi_n=\sum_{i=1}^{2^m} a_{ij}^{(m)} f_{2^m+i};$$

and let $\Phi=\{\varphi_n\colon n=0,1,\ldots\}$. The orthonormality of Φ follows from that of $\mathcal{F}$ and the orthogonality of each matrix A_m^T. To show that Φ is a Schauder basis for $C[0,1]$, one employs the following property of the foregoing matrices.

THEOREM B (OLEVSKII). *There is a positive constant C such that, for each $m=1,2,\ldots$, and every i and q, with $1\le i,q\le 2^m$,*

$$\sum_{k=1}^{2^m}\left|\sum_{j=1}^{q} a_{ij}^{(m)}a_{kj}^{(m)}\right|<C.$$

THEOREM 1. *Φ is a Schauder basis for $C[0,1]$ (and thus also for each $L^p[0,1]$, $1\le p<\infty$).*

PROOF. Because the linear span of Φ coincides with the linear span of the Franklin system and is, therefore, dense in $C[0,1]$, it suffices to show that the Lebesgue functions of Φ constitute a bounded set.

Let $n=2^m+q$, and let

$$K_n(x,t)=\sum_{r=0}^{n}\varphi_r(x)\varphi_r(t)$$

be the corresponding Dirichlet kernel. One has

$$\sum_{r=2^l+1}^{2^{l+1}} \varphi_r(x)\varphi_r(t) = \sum_{r=2^l+1}^{2^{l+1}} f_r(x)f_r(t),$$

by virtue of the orthogonality of the matrix A_l. It follows that $\sum_{r=0}^{2^m} \varphi_r(x)\varphi_r(t)$ is identical to the Dirichlet kernel of order 2^m associated with the Franklin system. Because the Franklin system is a Schauder basis for $C[0,1]$, there is a constant F, independent of m, such that

$$\int_0^1 \left|\sum_{r=0}^{2^m} \varphi_r(x)\varphi_r(t)\right| dt < F.$$

Moreover,

$$\begin{aligned}
&\int_0^1 \left|\sum_{r=2^m+1}^{n} \varphi_r(x)\varphi_r(t)\right| dt \\
&\quad = \int_0^1 \left|\sum_{j=1}^{q} \varphi_{2^m+j}(x)\varphi_{2^m+j}(t)\right| dt \\
&\quad = \int_0^1 \left|\sum_{j=1}^{q} \left(\sum_{i=1}^{2^m} a_{ij}^{(m)} f_{2^m+i}(x)\right)\left(\sum_{k=1}^{2^m} a_{jk}^{(m)} f_{2^m+k}(t)\right)\right| dt \\
&\quad \le \sum_{i=1}^{2^m} |f_{2^m+i}(x)| \int_0^1 \left|\sum_{k=1}^{2^m}\sum_{j=1}^{q} a_{ij}^{(m)} a_{kj}^{(m)} f_{2^m+k}(t)\right| dt \\
&\quad \le \sum_{i=1}^{2^m} |f_{2^m+i}(x)| \left\{\sum_{k=1}^{2^m} \left|\sum_{j=1}^{q} a_{ij}^{(m)} a_{kj}^{(m)}\right| \int_0^1 |f_{2^m+k}(t)|\,dt\right\} \\
&\quad \le \sum_{i=1}^{2^m} |f_{2^m+i}(x)| \left\{\sum_{k=1}^{2^m} \left|\sum_{j=1}^{q} a_{ij}^{(m)} a_{kj}^{(m)}\right|\right\} A2^{-m/2} \\
&\quad \le \sum_{i=1}^{2^m} \left|f_{2^m+i}(x)\right|\{C\}A2^{-m/2} \le CA^2,
\end{aligned}$$

by virtue of Theorems A and B. Finally,

$$\int_0^1 |K_n(x,t)|\,dt \le \int_0^1 \left|\sum_{r=0}^{2^m} \varphi_r(x)\varphi_r(t)\right| dt + \int_0^1 \left|\sum_{r=2^m+1}^{n} \varphi_r(x)\varphi_r(t)\right| dt \le F + CA^2,$$

for every natural number n.

3. The following proposition is the heart of the argument that demonstrates the conditionality of Φ as a basis for $L^p[0,1]$ when $1 < p < \infty$ and $p \neq 2$. (The author is indebted to the referee for the suggestion that Lemma 2 be substituted for the more cumbersome result that occupied this position in the original exposition of Theorem 3.)

LEMMA 2. *Let $N = 2^m$ with $m = 0, 1, \ldots$, let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence of real numbers, and let A be the positive constant introduced in Theorem* A. *Then, for each p in $[1, +\infty)$,*

$$A^{-1}\left(\frac{p}{2}+1\right)^{-1/2} N^{(1/2-1/p)}\left(\sum_{n=N+1}^{2N}|a_n|^p\right)^{1/p} \le \left\|\left(\sum_{n=N+1}^{2N} a_n^2 f_n^2\right)^{1/2}\right\|_p \le 8AN^{(1/2-1/p)}\left(\sum_{n=N+1}^{2N}|a_n|^p\right)^{1/p}.$$

PROOF. Let r_n denote the nth element of the Rademacher system. From an application of Theorem A, one obtains

$$A^{-1}N^{(1/2-1/p)}\left(\sum_{n=N+1}^{2N}|a_n|^p\right)^{1/p} \le \left\|\sum_{n=N+1}^{2N} r_n(t)a_n f_n\right\|_p \le AN^{(1/2-1/p)}\left(\sum_{n=N+1}^{2N}|a_n|^p\right)^{1/p}$$

for almost all t in $[0, 1]$, and, thus, it follows that

$$A^{-p}N^{p(1/2-1/p)}\sum_{n=N+1}^{2N}|a_n|^p \le \int_0^1\left\{\int_0^1\left|\sum_{n=N+1}^{2N} r_n(t)a_n f_n(x)\right|^p dx\right\} dt \le A^p N^{p(1/2-1/p)}\sum_{n=N+1}^{2N}|a_n|^p.$$

On the other hand, if one inverts the order of integration and then applies the Khinchin inequalities, one obtains

$$8^{-p}\int_0^1\left(\sum_{n=N+1}^{2N} a_n^2 f_n^2\right)^{p/2} dx \le \int_0^1\left\{\int_0^1\left|\sum_{n=N+1}^{2N} r_n(t)a_n f_n\right|^p dx\right\} dt \le \left(\frac{p}{2}+1\right)^{p/2}\int_0^1\left(\sum_{n=N+1}^{2N} a_n^2 f_n^2\right)^{p/2} dx,$$

from which the desideratum follows.

THEOREM 3. *Φ is not an unconditional basis for any of the spaces $L^p[0, 1]$ with $1 < p < \infty$ and $p \neq 2$.*

PROOF. By virtue of the standard results on interpolation and conjugacy, one need consider only the case $1 < p < 2$.

Let $g_m = f_{2^m+1}$, and let

$$a_n(g_m) = \int_0^1 g_m(t)\varphi_n(t)\, dt$$

be the nth Fourier coefficient in the expansion of g_m relative to Φ. According to a criterion established by Gaposhkin [**3**], were Φ an unconditional basis for $L^p[0,1]$, there would exist a positive constant G such that

$$G\left\|\left\{\sum_{n=1}^{\infty} a_n^2(g_m)\varphi_n^2\right\}^{1/2}\right\|_p \le \|g_m\|_p \le G^{-1}\left\|\left\{\sum_{n=1}^{\infty} a_n^2(g_n)\varphi_n^2\right\}^{1/2}\right\|_p .$$

Since

$$g_m = \sum_{j=1}^{2^m} 2^{-m/2}\varphi_{2^m+j},$$

one has

$$\sum_{n=1}^{\infty} a_n^2(g_m)\varphi_n^2 = 2^{-m}\sum_{j=1}^{2^m}\varphi_{2^m+j}^2 = 2^{-m}\sum_{j=1}^{2^m} f_{2^m+j}^2,$$

so that

$$\left\|\left\{\sum_{n=1}^{\infty} a_n^2(g_m)\varphi_n^2\right\}^{1/2}\right\|_p \ge A^{-1}\left(\frac{p}{2}+1\right)^{-1/2},$$

by virtue of Lemma 2. Upon combining this observation with the estimate for $\|g_m\|_p$, provided by Theorem A, one finds that

$$\rho_m = \|g_m\|_p \Big/ \left\|\left\{\sum_{n=1}^{\infty} a_n^2(g_m)\varphi_n^2\right\}^{1/2}\right\|_p \le A^2\left(\frac{p}{2}+1\right)^{1/2} 2^{m(1/2-1/p)}.$$

It follows that $\lim_m \rho_m = 0$, so that the Gaposhkin criterion cannot be fulfilled.

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