PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 1, May 1986

A NONLINEAR ERGODIC THEOREM FOR A REVERSIBLE SEMIGROUP OF NONEXPANSIVE MAPPINGS IN A HILBERT SPACE

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ABSTRACT. Let C be a nonempty closed convex subset of a Hilbert space, S a right reversible semitopological semigroup, $\mathcal{S} = \{T_t : t \in S\}$ a continuous representation of S as nonexpansive mappings on a closed convex subset C into C, and $F(\mathcal{S})$ the set of common fixed points of mappings T_t, $t \in S$. Then we deal with the existence of a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$ and Px is contained in the closure of the convex hull of $\{T_t x : t \in S\}$ for each $x \in C$.

1. Introduction. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed convex subset of H. A mapping $T: C \to C$ is called nonexpansive [**3**] on C, if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for every } x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T. The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [**1**]: Let C be a closed convex subset of a Hilbert space and T a nonexpansive mapping of C into itself. If the set $F(T)$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{co}}\{T^n x : n = 0, 1, 2, \ldots\}$ for each $x \in C$, where $\overline{\text{co}}\, A$ is the closure of the convex hull of A. In [**11**], the author proved the existence of such a retraction—"ergodic retraction"—for an amenable semigroup of nonexpansive mappings in a Hilbert space. Then, Hirano and Takahashi [**5**] extended this result to a Banach space. On the other hand, Rodé [**10**] found a sequence of means on the semigroups, generalizing the Cesàro means on positive integers, such that the corresponding sequence of mappings converges to a retraction onto the set of common fixed points. Recently Lau [**7**] considered the problem of weak convergence for a right reversible semigroup of nonexpansive mappings.

In this paper, we deal with the existence of "ergodic retraction" for a right reversible semigroup of nonexpansive mappings, that is, we prove a nonlinear ergodic theorem for such a semigroup in a Hilbert space. This theorem is a generalization

Received by the editors December 4, 1984 and, in revised form, May 1, 1985.
1980 *Mathematics Subject Classification*. Primary 47A35, 47H09.
Key words and phrases. Ergodic theorem, reversible semigroup, nonexpansive mapping, fixed point.

of the author's result [**11**]. In the proof, we also give a characterization of "ergodic retraction".

2. Nonlinear ergodic theorem. Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from S to S are continuous. S is called right reversible if any two closed left ideals of S have nonvoid intersection. In this case, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $a \leq b$ if and only if $\{a\} \cup \overline{Sa} \supseteq \{b\} \cup \overline{Sb}$, $a, b \in S$.

Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups; see [**4**, **6**, p. 335]. Let C be a closed convex subset of a real Hilbert space H and $\mathcal{S} = \{T_s : s \in S\}$ a continuous representation of S as nonexpansive mappings on a closed convex set C into C, i.e., $T_{ab}(x) = T_a T_b(x)$, $a, b, \in S$, $x \in C$ and the mapping $(s, x) \to T_s(x)$ from $S \times C$ into C is continuous when $S \times C$ has the product topology. Let $F(\mathcal{S})$ denote the set $\{x \in C : T_s x = x \text{ for all } s \in S\}$ of common fixed points of mappings T_s, $s \in S$ in C. Then, as is well known, $F(\mathcal{S})$ is a closed convex subset of C. Let Q be the metric projection of H onto $F(\mathcal{S})$. Then, by Phelps [**9**], Q is nonexpansive. Now we prove a nonlinear ergodic theorem for a right reversible semigroup of nonexpansive mappings in a Hilbert space.

THEOREM. *Let C be a nonempty closed convex subset of a real Hilbert space H, S a right reversible semitopological semigroup and $\mathcal{S} = \{T_s : s \in S\}$ a continuous representation of S as nonexpansive mappings of a closed convex set C into C. Suppose that*

$$F(\mathcal{S}) = \bigcap \{F(T_s) : s \in S\} \neq \emptyset.$$

Then the following are equivalent:

(a) $\bigcap_{s \in S} \overline{\text{co}}\{T_t x : t \geq s\} \cap F(\mathcal{S}) \neq \emptyset$ *for each* $x \in C$.

(b) *There is a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_t x : t \in S\}$ for every $x \in C$.*

PROOF. (b)$\Rightarrow$(a). Let $x \in C$. Then $Px \in F(\mathcal{S})$. Also

$$Px \in \bigcap_{s \in S} \overline{\text{co}}\{T_t x : t \geq s\}.$$

In fact,

$$Px = PT_s x \in \overline{\text{co}}\{T_t T_s x : t \in S\} \subset \overline{\text{co}}\{T_t x : t \geq s\}$$

for every $s \in S$.

(a)$\Rightarrow$(b). Let $x \in C$ and $f \in F(\mathcal{S})$. Let $b \geq a$. Then, since $b \in \{a\} \cup \overline{Sa}$, we may assume $b \in \overline{Sa}$. Let $\{s_\alpha\}$ be a net in S such that $s_\alpha a \to b$. Then, for each α,

$$\|T_{s_\alpha a} x - f\|^2 = \|T_{s_\alpha}(T_a x) - T_{s_\alpha} f\|^2 \leq \|T_a x - f\|^2.$$

Hence, $\|T_b x - f\|^2 \leq \|T_a x - f\|^2$. So the $\lim_s \|T_s x - f\|^2$ exists. Let

$$g(f) = \lim_s \|T_s x - f\|^2 \quad \text{for every } f \in F(\mathcal{S})$$

and

$$r = \inf\{g(f) : f \in F(\mathcal{S})\}.$$

Then, since the real-valued function g on $F(\mathcal{S})$ is convex and continuous and $g(f) \to \infty$ as $\|f\| \to \infty$, from [**2**, p. 79], there exists $f_0 \in F(\mathcal{S})$ with $g(f_0) = r$. Hence the set

$$M(x) = \{f \in F(\mathcal{S}) \colon g(f) = r\}$$

is nonempty. We now show that $M(x)$ consists of one point. In fact, let $f_0, f_1 \in M(x)$. Then using the parallelogram law, we obtain

$$\left\| \frac{f_0 - f_1}{2} \right\|^2 = \frac{\|T_s - f_0\|^2}{2} + \frac{\|T_s - f_1\|^2}{2} - \left\| T_s x - \frac{f_0 + f_1}{2} \right\|^2$$

for every $s \in S$. So we have

$$\left\| \frac{f_0 - f_1}{2} \right\|^2 = r - \lim_s \left\| T_s x - \frac{f_0 + f_1}{2} \right\|^2 \leq 0$$

and hence $f_0 = f_1$.

Let $M(x) = \{u\}$ and let Q be the metric projection of H onto $F(\mathcal{S})$. Then from [**7**] we know that $QT_s x$ converges strongly to some $z \in F(\mathcal{S})$. We show $u = z$. Let $a \leq b$. Then we may assume $b \in \overline{Sa}$. Let $\{s_\alpha\}$ be a net in S such that $s_\alpha a \to b$. Then for each α,

$$\|QT_a x - T_{s_\alpha a} x\|^2 = \|T_{s_\alpha} QT_a x - T_{s_\alpha} T_a x\|^2 \leq \|QT_a x - T_a x\|^2.$$

So, we have $\|QT_a x - T_b x\|^2 \leq \|QT_a x - T_a x\|^2$. Hence if $a \leq b$, then

$$\|QT_a x - T_b x\|^2 \leq \|QT_a x - T_a x\|^2 \leq \|f - T_a x\|^2$$

for every $f \in F(\mathcal{S})$. Therefore, for any $a \in S$, we have

$$g(QT_a x) = \lim_b \|T_b x - QT_a x\|^2 \leq \|T_a x - f\|^2$$

for every $f \in F(\mathcal{S})$ by above. Since g is continuous and $QT_a x$ converges strongly to $z \in F(\mathcal{S})$, we have $g(z) \leq \lim_a \|T_a x - f\|^2 = g(f)$ for every $f \in F(\mathcal{S})$. Then $u = z = \lim_t QT_t x$.

From (a), let $v \in F(\mathcal{S}) \cap \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x \colon t \geq s\}$. Then, since

$$\|u - v\|^2 = \|T_s x - v\|^2 - \|T_s x - u\|^2 - 2\langle u - v, T_s x - u\rangle$$

for every $s \in S$, we have

$$\|u - v\|^2 + 2\lim_s \langle u - v, T_s x - u\rangle = \lim_s \|T_s x - v\|^2 - \lim_s \|T_s x - u\|^2 \geq 0.$$

Let $\varepsilon > 0$. Then we have

$$2\lim_s \langle u - v, T_s x - u\rangle > -\|u - v\|^2 - \varepsilon.$$

Hence there exists $s_0 \in S$ such that

$$2\langle u - v, T_s x - u\rangle > -\|u - v\|^2 - \varepsilon$$

for every $s \geq s_0$. Since $v \in \overline{\mathrm{co}}\{T_t x \colon t \geq s_0\}$, we have

$$2\langle u - v, v - u\rangle \geq -\|u - v\|^2 - \varepsilon.$$

This inequality implies $\|u - v\|^2 \leq \varepsilon$. Since ε is arbitrary, we have $u = v$. Therefore

$$F(\mathcal{S}) \cap \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x \colon t \geq s\} = \{u\}.$$

Set $Px = \lim_t QT_t x$ for every $x \in C$. Then we have $T_s Px = Px$ and

$$PT_s x = \lim_t QT_t T_s x = \lim_t QT_{ts} x = Px$$

for every $s \in S$ and $x \in C$. From $\{Px\} = F(\mathcal{S}) \cap \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x \colon t \geq s\}$, it is obvious that $Px \in \overline{\mathrm{co}}\{T_s x \colon s \in S\}$ for each $x \in C$. Since

$$\|Px - Py\| = \lim_t \|QT_t x - QT_t y\| \leq \|x - y\|$$

for every $x, y \in C$, it follows that P is nonexpansive.

Lau and Takahashi [**8**] obtained an analogous result in a Banach space. However the Theorem is sharper than [**8**]. The following corollary was actually proved in [**11**].

COROLLARY. *Let C be a nonempty closed convex subset of a real Hilbert space H and $\mathcal{S}$ an amenable semigroup of nonexpansive mappings t of C into itself. Suppose that*

$$F(\mathcal{S}) = \bigcap \{F(t) \colon t \in \mathcal{S}\} \neq \emptyset.$$

Then, there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $Pt = tP = P$ for every $t \in \mathcal{S}$ and $Px \in \overline{\mathrm{co}}\{tx \colon t \in \mathcal{S}\}$ for every $x \in C$.

PROOF. Let μ be an invariant mean on $\mathcal{S}$ and $x \in C$. Then, since $F(\mathcal{S}) \neq \emptyset$, $\{tx \colon t \in \mathcal{S}\}$ is bounded and hence for each y in H, the real-valued function $t \mapsto \langle tx, y\rangle$ is bounded. Denote by $\mu_t\langle tx, y\rangle$ the value of μ at the function. Then this is linear and continuous in y. So by the Riesz theorem, there exists an $x_0 \in H$ such that $\mu_t\langle tx, y\rangle = \langle x_0, y\rangle$ for every $y \in H$ and the point x_0 is contained in $F(\mathcal{S}) \cap \bigcap_{s \in S} \overline{\mathrm{co}}\{tx \colon t \geq s\}$. For details, see [**11**].

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