

A NONLINEAR ERGODIC THEOREM FOR A REVERSIBLE SEMIGROUP OF NONEXPANSIVE MAPPINGS IN A HILBERT SPACE

WATARU TAKAHASHI

ABSTRACT. Let C be a nonempty closed convex subset of a Hilbert space, S a right reversible semitopological semigroup, $\mathcal{S} = \{T_t : t \in S\}$ a continuous representation of S as nonexpansive mappings on a closed convex subset C into C , and $F(\mathcal{S})$ the set of common fixed points of mappings T_t , $t \in S$. Then we deal with the existence of a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and Px is contained in the closure of the convex hull of $\{T_t x : t \in S\}$ for each $x \in C$.

1. Introduction. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed convex subset of H . A mapping $T: C \rightarrow C$ is called nonexpansive [3] on C , if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for every } x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T . The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1]: Let C be a closed convex subset of a Hilbert space and T a nonexpansive mapping of C into itself. If the set $F(T)$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{co}}\{T^n x : n = 0, 1, 2, \dots\}$ for each $x \in C$, where $\overline{\text{co}}A$ is the closure of the convex hull of A . In [11], the author proved the existence of such a retraction—“ergodic retraction”—for an amenable semigroup of nonexpansive mappings in a Hilbert space. Then, Hirano and Takahashi [5] extended this result to a Banach space. On the other hand, Rodé [10] found a sequence of means on the semigroups, generalizing the Cesàro means on positive integers, such that the corresponding sequence of mappings converges to a retraction onto the set of common fixed points. Recently Lau [7] considered the problem of weak convergence for a right reversible semigroup of nonexpansive mappings.

In this paper, we deal with the existence of “ergodic retraction” for a right reversible semigroup of nonexpansive mappings, that is, we prove a nonlinear ergodic theorem for such a semigroup in a Hilbert space. This theorem is a generalization

Received by the editors December 4, 1984 and, in revised form, May 1, 1985.

1980 *Mathematics Subject Classification.* Primary 47A35, 47H09.

Key words and phrases. Ergodic theorem, reversible semigroup, nonexpansive mapping, fixed point.

of the author’s result [11]. In the proof, we also give a characterization of “ergodic retraction”.

2. Nonlinear ergodic theorem. Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \rightarrow a \cdot s$ and $s \rightarrow s \cdot a$ from S to S are continuous. S is called right reversible if any two closed left ideals of S have nonvoid intersection. In this case, (S, \leq) is a directed system when the binary relation “ \leq ” on S is defined by $a \leq b$ if and only if $\{a\} \cup \overline{Sa} \supseteq \{b\} \cup \overline{Sb}$, $a, b \in S$.

Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups; see [4, 6, p. 335]. Let C be a closed convex subset of a real Hilbert space H and $\mathcal{S} = \{T_s : s \in S\}$ a continuous representation of S as nonexpansive mappings on a closed convex set C into C , i.e., $T_{ab}(x) = T_a T_b(x)$, $a, b, \in S$, $x \in C$ and the mapping $(s, x) \rightarrow T_s(x)$ from $S \times C$ into C is continuous when $S \times C$ has the product topology. Let $F(\mathcal{S})$ denote the set $\{x \in C : T_s x = x \text{ for all } s \in S\}$ of common fixed points of mappings T_s , $s \in S$ in C . Then, as is well known, $F(\mathcal{S})$ is a closed convex subset of C . Let Q be the metric projection of H onto $F(\mathcal{S})$. Then, by Phelps [9], Q is nonexpansive. Now we prove a nonlinear ergodic theorem for a right reversible semigroup of nonexpansive mappings in a Hilbert space.

THEOREM. *Let C be a nonempty closed convex subset of a real Hilbert space H , S a right reversible semitopological semigroup and $\mathcal{S} = \{T_s : s \in S\}$ a continuous representation of S as nonexpansive mappings of a closed convex set C into C . Suppose that*

$$F(\mathcal{S}) = \bigcap \{F(T_s) : s \in S\} \neq \emptyset.$$

Then the following are equivalent:

- (a) $\bigcap_{s \in S} \overline{\text{co}}\{T_t x : t \geq s\} \cap F(\mathcal{S}) \neq \emptyset$ for each $x \in C$.
- (b) *There is a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_t x : t \in S\}$ for every $x \in C$.*

PROOF. (b) \Rightarrow (a). Let $x \in C$. Then $Px \in F(\mathcal{S})$. Also

$$Px \in \bigcap_{s \in S} \overline{\text{co}}\{T_t x : t \geq s\}.$$

In fact,

$$Px = PT_s x \in \overline{\text{co}}\{T_t T_s x : t \in S\} \subset \overline{\text{co}}\{T_t x : t \geq s\}$$

for every $s \in S$.

(a) \Rightarrow (b). Let $x \in C$ and $f \in F(\mathcal{S})$. Let $b \geq a$. Then, since $b \in \{a\} \cup \overline{Sa}$, we may assume $b \in \overline{Sa}$. Let $\{s_\alpha\}$ be a net in S such that $s_\alpha a \rightarrow b$. Then, for each α ,

$$\|T_{s_\alpha} a x - f\|^2 = \|T_{s_\alpha}(T_a x) - T_{s_\alpha} f\|^2 \leq \|T_a x - f\|^2.$$

Hence, $\|T_b x - f\|^2 \leq \|T_a x - f\|^2$. So the $\lim_s \|T_s x - f\|^2$ exists. Let

$$g(f) = \lim_s \|T_s x - f\|^2 \quad \text{for every } f \in F(\mathcal{S})$$

and

$$r = \inf\{g(f) : f \in F(\mathcal{S})\}.$$

Then, since the real-valued function g on $F(S)$ is convex and continuous and $g(f) \rightarrow \infty$ as $\|f\| \rightarrow \infty$, from [2, p. 79], there exists $f_0 \in F(S)$ with $g(f_0) = r$. Hence the set

$$M(x) = \{f \in F(S) : g(f) = r\}$$

is nonempty. We now show that $M(x)$ consists of one point. In fact, let $f_0, f_1 \in M(x)$. Then using the parallelogram law, we obtain

$$\left\| \frac{f_0 - f_1}{2} \right\|^2 = \frac{\|T_s - f_0\|^2}{2} + \frac{\|T_s - f_1\|^2}{2} - \left\| T_s x - \frac{f_0 + f_1}{2} \right\|^2$$

for every $s \in S$. So we have

$$\left\| \frac{f_0 - f_1}{2} \right\|^2 = r - \lim_s \left\| T_s x - \frac{f_0 + f_1}{2} \right\|^2 \leq 0$$

and hence $f_0 = f_1$.

Let $M(x) = \{u\}$ and let Q be the metric projection of H onto $F(S)$. Then from [7] we know that $QT_s x$ converges strongly to some $z \in F(S)$. We show $u = z$. Let $a \leq b$. Then we may assume $b \in \overline{S}a$. Let $\{s_\alpha\}$ be a net in S such that $s_\alpha a \rightarrow b$. Then for each α ,

$$\|QT_a x - T_{s_\alpha a} x\|^2 = \|T_{s_\alpha} QT_a x - T_{s_\alpha} T_a x\|^2 \leq \|QT_a x - T_a x\|^2.$$

So, we have $\|QT_a x - T_b x\|^2 \leq \|QT_a x - T_a x\|^2$. Hence if $a \leq b$, then

$$\|QT_a x - T_b x\|^2 \leq \|QT_a x - T_a x\|^2 \leq \|f - T_a x\|^2$$

for every $f \in F(S)$. Therefore, for any $a \in S$, we have

$$g(QT_a x) = \lim_b \|T_b x - QT_a x\|^2 \leq \|T_a x - f\|^2$$

for every $f \in F(S)$ by above. Since g is continuous and $QT_a x$ converges strongly to $z \in F(S)$, we have $g(z) \leq \lim_a \|T_a x - f\|^2 = g(f)$ for every $f \in F(S)$. Then $u = z = \lim_t QT_t x$.

From (a), let $v \in F(S) \cap \bigcap_{s \in S} \overline{\text{co}}\{T_t x : t \geq s\}$. Then, since

$$\|u - v\|^2 = \|T_s x - v\|^2 - \|T_s x - u\|^2 - 2\langle u - v, T_s x - u \rangle$$

for every $s \in S$, we have

$$\|u - v\|^2 + 2 \lim_s \langle u - v, T_s x - u \rangle = \lim_s \|T_s x - v\|^2 - \lim_s \|T_s x - u\|^2 \geq 0.$$

Let $\varepsilon > 0$. Then we have

$$2 \lim_s \langle u - v, T_s x - u \rangle > -\|u - v\|^2 - \varepsilon.$$

Hence there exists $s_0 \in S$ such that

$$2\langle u - v, T_s x - u \rangle > -\|u - v\|^2 - \varepsilon$$

for every $s \geq s_0$. Since $v \in \overline{\text{co}}\{T_t x : t \geq s_0\}$, we have

$$2\langle u - v, v - u \rangle \geq -\|u - v\|^2 - \varepsilon.$$

This inequality implies $\|u - v\|^2 \leq \varepsilon$. Since ε is arbitrary, we have $u = v$. Therefore

$$F(S) \cap \bigcap_{s \in S} \overline{\text{co}}\{T_t x : t \geq s\} = \{u\}.$$

Set $Px = \lim_t QT_t x$ for every $x \in C$. Then we have $T_s Px = Px$ and

$$PT_s x = \lim_t QT_t T_s x = \lim_t QT_{ts} x = Px$$

for every $s \in S$ and $x \in C$. From $\{Px\} = F(S) \cap \bigcap_{s \in S} \overline{\text{co}}\{T_t x : t \geq s\}$, it is obvious that $Px \in \overline{\text{co}}\{T_s x : s \in S\}$ for each $x \in C$. Since

$$\|Px - Py\| = \lim_t \|QT_t x - QT_t y\| \leq \|x - y\|$$

for every $x, y \in C$, it follows that P is nonexpansive.

Lau and Takahashi [8] obtained an analogous result in a Banach space. However the Theorem is sharper than [8]. The following corollary was actually proved in [11].

COROLLARY. *Let C be a nonempty closed convex subset of a real Hilbert space H and S an amenable semigroup of nonexpansive mappings t of C into itself. Suppose that*

$$F(S) = \bigcap \{F(t) : t \in S\} \neq \emptyset.$$

Then, there exists a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{tx : t \in S\}$ for every $x \in C$.

PROOF. Let μ be an invariant mean on S and $x \in C$. Then, since $F(S) \neq \emptyset$, $\{tx : t \in S\}$ is bounded and hence for each y in H , the real-valued function $t \mapsto \langle tx, y \rangle$ is bounded. Denote by $\mu_t \langle tx, y \rangle$ the value of μ at the function. Then this is linear and continuous in y . So by the Riesz theorem, there exists an $x_0 \in H$ such that $\mu_t \langle tx, y \rangle = \langle x_0, y \rangle$ for every $y \in H$ and the point x_0 is contained in $F(S) \cap \bigcap_{s \in S} \overline{\text{co}}\{tx : t \geq s\}$. For details, see [11].

REFERENCES

1. J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. A–B **280** (1975), 1511–1514.
2. V. Barbu and Th. Precupanu, *Convexity and optimization in Banach spaces*, Editura Academiei R.S.T., Bucharest, 1978.
3. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044.
4. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
5. N. Hirano and W. Takahashi, *Nonlinear ergodic theorems for an amenable semigroup of nonexpansive mappings in a Banach space*, Pacific J. Math. **112** (1984), 333–346.
6. R. D. Holmes and A. T. Lau, *Nonexpansive actions of topological semigroups and fixed points*, J. London Math. Soc. (2) **5** (1972), 330–336.
7. A. T. Lau, *Semigroup of nonexpansive mappings on a Hilbert space*, J. Math. Anal. Appl. **105** (1985), 514–522.
8. A. T. Lau and W. Takahashi, *Weak convergence and nonlinear ergodic theorems for reversible semigroups of nonexpansive mappings* (to appear).
9. R. R. Phelps, *Convex sets and nearest points*, Proc. Amer. Math. Soc. **8** (1957), 790–797.
10. G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl. **85** (1982), 172–178.
11. W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

DEPARTMENT OF INFORMATION SCIENCES, TOKYO INSTITUTE OF TECHNOLOGY, OH-KAYAMA, MEGURO-KU, TOKYO, JAPAN