

## A PROPERTY OF THE EMBEDDING OF $c_0$ IN $l_\infty$ <sup>1</sup>

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ABSTRACT. This note proves that if  $X$  is an FK space containing  $\{\delta^n\}$  and if  $X + c_0 = l_\infty$ , then  $X = l_\infty$ . The result is stronger than the fact that  $c_0$  is not complemented in  $l_\infty$ , and shows that separability can be dropped in a similar theorem of Bennett and Kalton. The proof depends on Schur's theorem and the fact that  $l_\infty$  is a GB space to show that  $X$  must be barrelled in  $l_\infty$ .

It is well known that the space  $c_0$  of null sequences is not complemented in the bounded sequences  $l_\infty$ . This note will show that if  $X$  is an FK space containing  $\phi$ , the space of finitely nonzero sequences, and if  $X + c_0 = l_\infty$ , then  $X = l_\infty$ . The latter is stronger than the fact that  $c_0$  is uncomplemented.

Bennett and Kalton in [1, Theorems 24, 25] proved that if  $X$  is a separable FK space containing  $\phi$  and if  $X + c_0 \supset l_\infty$ , then  $X \supset l_\infty$ . The present note shows that separability can be dropped in the Bennett and Kalton theorem. Their technique involved a complicated construction to show that two topologies on a particular space yield the same convergent sequences and hence the same compact sets. Separability of  $X$  allows an application of the Kalton closed graph theorem of [3]. The enclosed proof is based on Schur's theorem and the fact that weak and weak\* sequential convergence coincide in  $l'_\infty$ . The latter establishes that  $X \cap l_\infty$  is barrelled in  $l_\infty$ .

It is possible to provide a systematic treatment of the embedding property exhibited by this work. However, the theorem and technique of proof seem sufficiently different from the general thread and of sufficient interest to justify an independent treatment.

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An FK space is a locally convex complete linear metric space of real or complex sequences. The required properties of FK spaces may be found in the first several chapters of [6]. If  $X$  and  $Y$  are FK spaces with seminorms  $\{p_n\}$ ,  $\{q_n\}$ , respectively, then  $X + Y$  is an FK space with seminorms  $\{r_{nm}\}$  given by

$$r_{nm}(z) = \inf\{p_n(x) + q_m(y) : z = x + y, x \in X, y \in Y\}.$$

**THEOREM 1.** *If  $X$  is an FK space containing  $\phi$  and if  $X + c_0 = l_\infty$ , then  $X = l_\infty$ .*

**PROOF.** Since  $\phi \subset X$ , the closure of  $X$  in  $l_\infty$  contains  $c_0$ , so  $X$  must be dense in  $l_\infty$ .

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Let  $\{\mu_n\} \subset l'_\infty$  with  $\mu_n \rightarrow 0$  in the topology  $\sigma(l'_\infty, X)$ . The proof is completed by establishing that  $\mu_n \rightarrow 0$  in  $\sigma(l'_\infty, l_\infty)$ . Then  $X$  must be barrelled in  $l_\infty$ , and the result of Bennett and Kalton [2, p. 514] applies, showing that  $X = l_\infty$ .

Every  $\mu \in l'_\infty$  can be written uniquely in the form  $\mu = \lambda + \nu$  where  $\lambda(w) = \sum \lambda_k w_k$ ,  $w \in l_\infty$ , and  $\nu$  vanishes on  $c_0$ . Then  $A\mu = \lambda$  defines a bounded operator on  $l'_\infty$  which may be considered an operator into  $l_1$ . Let  $\lambda^n = A(\mu_n)$ .

It suffices to prove that  $\lambda^n \rightarrow 0$  in  $l_1$ . To see this, let  $w \in l_\infty$  and write  $w = x + a$  where  $x \in X$ ,  $a \in c_0$ . Then

$$\mu_n(w) = \mu_n(x) + \sum_k \lambda_k^n a_k,$$

since  $\mu_n - A\mu_n$  vanishes on  $c_0$ , so  $\mu_n \rightarrow 0$  in  $\sigma(l'_\infty, l_\infty)$ .

Assume that  $t_n = \|\lambda^n\|_1 \geq \varepsilon > 0$  for all  $n$ . As in the previous paragraph,  $\{(1/t_n)\mu_n\}$  is bounded in  $l'_\infty$ . Since  $X$  is dense in  $l_\infty$ ,  $(1/t_n)\mu_n \rightarrow 0$  in  $\sigma(l'_\infty, l_\infty)$ . Since  $l_\infty$  is a GB space by Grothendieck's theorem (see [5, 14-7-7]), it follows that  $(1/t_n)\mu_n \rightarrow 0$  in  $\sigma(l'_\infty, l''_\infty)$ . Now  $A: l'_\infty \rightarrow l_1$  is weakly continuous, so  $(1/t_n)\lambda^n \rightarrow 0$  in  $\sigma(l_1, l_\infty)$ . The latter contradicts Schur's theorem, since  $\|(1/t_n)\lambda^n\|_1 = 1$ .

Therefore,  $\lambda^n \rightarrow 0$  in  $l_1$  and the proof is complete.  $\square$

It is easy to see that Theorem 1 is stronger than the assertion that  $c_0$  is not complemented in  $l_\infty$ . For convenience the notation  $Y < Z$  means that  $X = Z$  whenever  $X$  is an FK space containing  $\phi$  and  $X + Y = Z$ .

**THEOREM 2.** (i) *If  $Y$  is a closed complemented subspace of  $l_\infty$  containing  $c_0$ , then  $Y \not< l_\infty$ .*

(ii) *There exists a closed uncomplemented subspace  $Y \supset c_0$  of  $l_\infty$  such that  $Y \not< l_\infty$ .*

**PROOF.** If  $W$  is a closed subspace of  $l_\infty$  with  $Y \oplus W = l_\infty$ , then take  $X = W \oplus l_1$  to show that  $Y \not< l_\infty$ .

To prove (ii), let  $\{S_n\}$  be a partition of the positive integers into infinite subsets, and write each  $S_n$  as an increasing sequence  $\{r_k^n\}$ . Let

$$Y = \left\{ y \in l_\infty : \lim_k y_{r_k^n} = 0 \text{ for each } n \right\}.$$

Then  $Y$  is not isomorphic to  $l_\infty$ , since  $c_0$  is a quotient of  $Y$ . (See [5, 14-7-8].) As shown in [4],  $Y$  is not complemented in  $l_\infty$ . To see that  $Y \not< l_\infty$ , just take

$$X = \left\{ x \in l_\infty : \lim_n x_{r_1^n} = 0 \right\}. \quad \square$$

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