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A PROPERTY OF THE EMBEDDING OF c_0 IN l_∞[1]

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ABSTRACT. This note proves that if X is an FK space containing $\{\delta^n\}$ and if $X + c_0 = l_\infty$, then $X = l_\infty$. The result is stronger than the fact that c_0 is not complemented in l_∞, and shows that separability can be dropped in a similar theorem of Bennett and Kalton. The proof depends on Schur's theorem and the fact that l_∞ is a GB space to show that X must be barrelled in l_∞.

It is well known that the space c_0 of null sequences is not complemented in the bounded sequences l_∞. This note will show that *if X is an FK space containing ϕ, the space of finitely nonzero sequences, and if $X + c_0 = l_\infty$, then $X = l_\infty$*. The latter is stronger than the fact that c_0 is uncomplemented.

Bennett and Kalton in [**1**, Theorems 24, 25] proved that if X is a separable FK space containing ϕ and if $X + c_0 \supset l_\infty$, then $X \supset l_\infty$. The present note shows that separability can be dropped in the Bennett and Kalton theorem. Their technique involved a complicated construction to show that two topologies on a particular space yield the same convergent sequences and hence the same compact sets. Separability of X allows an application of the Kalton closed graph theorem of [**3**]. The enclosed proof is based on Schur's theorem and the fact that weak and weak* sequential convergence coincide in l'_∞. The latter establishes that $X \cap l_\infty$ is barrelled in l_∞.

It is possible to provide a systematic treatment of the embedding property exhibited by this work. However, the theorem and technique of proof seem sufficiently different from the general thread and of sufficient interest to justify an independent treatment.

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An *FK space* is a locally convex complete linear metric space of real or complex sequences. The required properties of FK spaces may be found in the first several chapters of [**6**]. If X and Y are FK spaces with seminorms $\{p_n\}$, $\{q_n\}$, respectively, then $X + Y$ is an FK space with seminorms $\{r_{nm}\}$ given by

$$r_{nm}(z) = \inf\{p_n(x) + q_m(y) \colon z = x + y, x \in X, y \in Y\}.$$

THEOREM 1. *If X is an FK space containing ϕ and if $X + c_0 = l_\infty$, then $X = l_\infty$.*

PROOF. Since $\phi \subset X$, the closure of X in l_∞ contains c_0, so X must be dense in l_∞.

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Let $\{\mu_n\} \subset l'_\infty$ with $\mu_n \to 0$ in the topology $\sigma(l'_\infty, X)$. The proof is completed by establishing that $\mu_n \to 0$ in $\sigma(l'_\infty, l_\infty)$. Then X must be barrelled in l_∞, and the result of Bennett and Kalton [**2**, p. 514] applies, showing that $X = l_\infty$.

Every $\mu \in l'_\infty$ can be written uniquely in the form $\mu = \lambda + \nu$ where $\lambda(w) = \sum \lambda_k w_k$, $w \in l_\infty$, and ν vanishes on c_0. Then $A\mu = \lambda$ defines a bounded operator on l'_∞ which may be considered an operator into l_1. Let $\lambda^n = A(\mu_n)$.

It suffices to prove that $\lambda^n \to 0$ in l_1. To see this, let $w \in l_\infty$ and write $w = x + a$ where $x \in X$, $a \in c_0$. Then

$$\mu_n(w) = \mu_n(x) + \sum_k \lambda_k^n a_k,$$

since $\mu_n - A\mu_n$ vanishes on c_0, so $\mu_n \to 0$ in $\sigma(l'_\infty, l_\infty)$.

Assume that $t_n = \|\lambda^n\|_1 \geq \varepsilon > 0$ for all n. As in the previous paragraph, $\{(1/t_n)\mu_n\}$ is bounded in l'_∞. Since X is dense in l_∞, $(1/t_n)\mu_n \to 0$ in $\sigma(l'_\infty, l_\infty)$. Since l_∞ is a GB space by Grothendieck's theorem (see [**5**, 14-7-7]), it follows that $(1/t_n)\mu_n \to 0$ in $\sigma(l'_\infty, l''_\infty)$. Now $A: l'_\infty \to l_1$ is weakly continuous, so $(1/t_n)\lambda^n \to 0$ in $\sigma(l_1, l_\infty)$. The latter contradicts Schur's theorem, since $\|(1/t_n)\lambda^n\|_1 = 1$.

Therefore, $\lambda^n \to 0$ in l_1 and the proof is complete. □

It is easy to see that Theorem 1 is stronger than the assertion that c_0 is not complemented in l_∞. For convenience the notation $Y < Z$ means that $X = Z$ whenever X is an FK space containing ϕ and $X + Y = Z$.

THEOREM 2. (i) *If Y is a closed complemented subspace of l_∞ containing c_0, then $Y \not< l_\infty$.*

(ii) *There exists a closed uncomplemented subspace $Y \supset c_0$ of l_∞ such that $Y \not< l_\infty$.*

PROOF. If W is a closed subspace of l_∞ with $Y \oplus W = l_\infty$, then take $X = W \oplus l_1$ to show that $Y \not< l_\infty$.

To prove (ii), let $\{S_n\}$ be a partition of the positive integers into infinite subsets, and write each S_n as an increasing sequence $\{r_k^n\}$. Let

$$Y = \left\{ y \in l_\infty : \lim_k y_{r_k^n} = 0 \text{ for each } n \right\}.$$

Then Y is not isomorphic to l_∞, since c_0 is a quotient of Y. (See [**5**, 14-7-8].) As shown in [**4**], Y is not complemented in l_∞. To see that $Y \not< l_\infty$, just take

$$X = \left\{ x \in l_\infty : \lim_n x_{r_1^n} = 0 \right\}. \quad \square$$

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