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SECOND ORDER DIFFERENTIAL EQUATIONS WITH TRANSCENDENTAL COEFFICIENTS

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ABSTRACT. Let w_1 and w_2 be two linearly independent solutions to $w'' + Aw = 0$, where A is a transcendental entire function of order $\rho(A) < 1$. We show that the exponent of convergence $\lambda(E)$ of the zeros of $E = w_1 w_2$ is either infinite or satisfies $\rho(A)^{-1} + \lambda(E)^{-1} \leq 2$. For $\rho(A) = \frac{1}{2}$, this answers a question of Bank.

1. Introduction. In this paper we prove

THEOREM 1. *Let A be a transcendental entire function of order $\rho(A) < 1$. If w_1 and w_2 are two linearly independent entire solutions to*

$$w'' + Aw = 0, \tag{1.1}$$

then the exponent of convergence $\lambda(E)$ of the sequence of zeros of $E = w_1 w_2$ is either infinite or satisfies

$$\rho(A)^{-1} + \lambda(E)^{-1} \leq 2. \tag{1.2}$$

Specifically,

$$\rho(A) \leq \tfrac{1}{2} \Rightarrow \lambda(E) = +\infty. \tag{1.3}$$

In particular, the implication (1.3) answers a question posed to us by Bank. In [**2**] he and Laine proved that $\rho(A) < \frac{1}{2}$ implies $\lambda(E) = +\infty$. Their method, which combines the $\cos \pi\rho$ theorem and Wiman–Valiron theory, does not seem to cover the case $\rho = \frac{1}{2}$.

To prove Theorem 1, we develop another method based on the Beurling–Tsuji estimate for harmonic measure [**6**, p. 116]. The idea of using this estimate was suggested to us by an unpublished manuscript of Edrei in which he attacks a similar problem. By applying a related technique, the spread theorem [**1**], he proves that (2.1) can never hold if both $A(z)$ and $E(z)$ have orders less than one and $A(z)$ satisfies a strong regularity condition.

In an earlier unpublished version of this paper, the author proved (1.3) using the Beurling–Tsuji inequality (together with a modification of some regularity theorems concerning functions extremal for the $\cos \pi\rho$ theorem). Using the same inequality, L. C. Shen [**5**] proved independently that $\rho(A) < 1$ implies $\lambda(E) \geq 1$. Theorem 1 generalizes both results.

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Bank has pointed out to us that Theorem 1 has a curious

COROLLARY 1. *There does not exist an entire function E, $\rho(E) < 1$, such that the value of $E'(z)$ at every zero of $E(z)$ is ± 1.*

This result is contained in the work of Shen [**5**] and should be credited to him. It follows since by Lemma C in [**3**] such a function would have to be the product of two linearly independent solutions of (1.1) where, by (2.3), $\rho(A) < 1$. Theorem 1 then says $\lambda(E) > 1$ which implies that $\rho(E) > 1$, a contradiction. We note that for $\rho(E) = \frac{1}{2}$, $E(z) = 2\sqrt{z}\sin\sqrt{z}$ almost provides a counterexample to Corollary 1 except at zero, where $E'(0) = 2$.

It is an open question whether (1.3) holds provided $\rho(A) < 1$ or whether (1.2) is sharp. More generally it is not known whether (1.3) holds if $\rho(A)$ does not equal an integer. It certainly does not hold if $\rho(A)$ equals an integer or infinity, [**2**, §5b].

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2. Preliminaries. Let w_1 and w_2 be linearly independent entire solutions to (1.1). Set $E = w_1 w_2$. Then, as in [**2**, p. 354],

$$4A = (E'/E)^2 - 2(E''/E) - c/E^2, \tag{2.1}$$

where c is the (constant) Wronskian of w_1 and w_2. We assume from now on that the zeros of E have finite exponent of convergence and show that (2.1) implies (1.2).

We assume familiarity with some basic Nevanlinna theory. Thus, by (2.1) and the lemma of the logarithmic derivative, we have

$$m(r, 1/E) = \tfrac{1}{2}m(r, A) + O(\log(rT(r, E))), \tag{2.2}$$

where $r \to \infty$ outside a set G of finite measure. Adding $N(r, 1/E)$ to both sides of (2.2), and appealing to the first fundamental theorem of Nevanlinna theory and the fact that A and E are entire, gives

$$T(r, E) = N(r, 1/E) + \tfrac{1}{2}T(r, A) + O(\log(rT(r, E))) \tag{2.3}$$

as $r \to \infty$, $r \notin G$. It follows that we need only prove

$$\rho(A)^{-1} + \rho(E)^{-1} \leq 2, \tag{2.4}$$

for this will imply (1.2). Indeed if (2.4) holds, then $\rho(A) < \rho(E)$ since $\rho(A) < 1$. Thus, $\lambda(E) = \rho(E)$ by (2.3).

Since the zeros of E have finite exponent of convergence and A has finite order, it also follows from (2.3) that the order of E, $\rho(E) < \infty$. Thus we have

LEMMA 1. *Given $\varepsilon > 0$ there exists $C = C(\varepsilon)$ such that*

$$|(E'/E)^2(re^{i\theta}) - (2E''/E)(re^{i\theta})| \leq r^C \tag{2.5}$$

for all $r \geq r_0 > 1$ and all $\theta \notin J(r)$, where the angular measure of $J(r)$, $m(J(r)) \leq \varepsilon\pi$.

PROOF. Let $H = (E'/E)^2(re^{i\theta}) - (2E''/E)(re^{i\theta})$. Clearly

$$m(r, H) \leq m(r, E''/E') + 3m(r, E'/E) + O(1).$$

Thus, by the lemma of the logarithmic derivative and the fact that E (and E') have finite order, there exists a constant K such that $m(r, H) \leq K \log r$ for all $r \geq r_0$. Now fix $\varepsilon > 0$ and let $C = 2K/\varepsilon$. Then by the definition of $m(r, H)$ and K we easily have that for every $r \geq r_0$, the set $J(r)$ of θ where $|H(re^{i\theta})| \geq r^C$ has angular measure at most $\varepsilon\pi$. The proof is complete.

To state the next lemma we need some notation. Let D be a region in $\mathbf{C}$. To each $r \in \mathbf{R}^+$ set $\theta_D^*(r) = \theta^*(r) = +\infty$ if the entire circle $|z| = r$ lies in D. Otherwise, let $\theta_D^*(r) = \theta^*(r)$ be the measure of all θ in $[0, 2\pi)$ such that $re^{i\theta} \in D$. As usual, the order $\rho(u)$ of a function u subharmonic in the plane is given by $\rho(u) = \overline{\lim}_{r\to\infty} \log M(r, u)/\log r$, where $M(r, u)$ is the maximum modulus of u on a circle of radius r.

LEMMA 2. *Let u be a subharmonic function in $\mathbf{C}$ and let D be an open component of $\{z : u(z) > 0\}$. Then*

$$\rho(u) \geq \overline{\lim_{R\to\infty}} (\log R)^{-1}\pi \int_1^R dt/t\theta_D^*(t). \tag{2.6}$$

Furthermore, given $\varepsilon > 0$, define $F = \{r : \theta_D^*(r) \leq \varepsilon\pi\}$. Then

$$\overline{\lim_{R\to\infty}} (\log R)^{-1} \int_{F\cap[1,R]} dt/t \leq \varepsilon\rho(u). \tag{2.7}$$

PROOF. An easy application of the maximum principle gives us u unbounded in D. Thus we can find a point $z_0 \in D$ such that $u(z_0) \geq 1$. By the Beurling–Tsuji inequality [**6**, p. 116] we obtain for $R \geq 4|z_0|$ that

$$\log M(R, u) \geq \pi \int_{2|z_0|}^{R/2} dt/t\theta_D^*(t) + C_0, \tag{2.8}$$

where C_0 is an absolute constant. Clearly (2.6) follows from (2.8). Also by (2.8) and the definition of F we obtain

$$\log M(R, u) \geq \varepsilon^{-1} \int_{F\cap[2|z_0|, R/2]} dt/t + C_0, \tag{2.9}$$

from which (2.7) follows. The proof is complete.

We need one more lemma whose proof can be found in [**4**, 676]. We prove it for completeness.

LEMMA 3. *Let $l_1(t) > 0$, $l_2(t) > 0$ $(t \geq t_0)$ be two measurable functions on $(0, \infty)$ with $l_1(t) + l_2(t) \leq (2 + \varepsilon)\pi$, where $\varepsilon > 0$. If $G \subseteq (0, \infty)$ is any measurable set and*

$$\pi \int_G dt/tl_1(t) \leq \alpha \int_G dt/t, \qquad \alpha \geq 1/2, \tag{2.10}$$

then

$$\pi \int_G dt/tl_2(t) \geq \frac{\alpha}{(2+\varepsilon)\alpha - 1} \int_G dt/t. \tag{2.11}$$

PROOF. By the Cauchy–Schwarz inequality

$$\int_G dt/tl_j(t) \int_G l_j(t)\, dt/t \geq \left(\int_G dt/t\right)^2, \qquad j = 1, 2. \tag{2.12}$$

From (2.10) and (2.12) with $j = 1$ we obtain

$$\int_G l_1(t)\,dt/t \geq \frac{\pi}{\alpha}\int_G dt/t. \tag{2.13}$$

Thus

$$\begin{aligned}\int_G l_2(t)\,dt/t &\leq \int_G [(2+\varepsilon)\pi - l_1(t)]\,dt/t \\ &\leq \left[(2+\varepsilon)\pi - \frac{\pi}{\alpha}\right]\int_G dt/t.\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.12) with $j = 2$ gives (2.11).

3. Proof of Theorem 1. Fix $\varepsilon > 0$ and let N be an integer such that

$$N > C = C(\varepsilon), \tag{3.1}$$

where C is as in Lemma 1 and

$$\log M(2, A) < N \log 2. \tag{3.2}$$

Since A is transcendental there exists z_0, $|z_0| > 2$, such that $\log|A(z_0)| > N \log|z_0|$. Let D_1 be the component of the set

$$\{z\colon \log|A(z)| - N\log|z| > 0\}$$

containing z_0. Clearly D_1 is open and since (3.2) holds, $\log|A(z)| - N\log|z|$ is subharmonic in D_1 and identically zero on ∂D_1. Thus, if we define

$$u(z) = \begin{cases} \log|A(z)| - N\log|z|, & z \in D, \\ 0, & z \in \mathbf{C}\setminus D,\end{cases}$$

we have that $u(z)$ is subharmonic in $\mathbf{C}$ with

$$\rho(u) \leq \rho(A). \tag{3.3}$$

Let D_2 be any component of $\{z\colon \log|E(z)| > 0\}$ and let $D_3 = \{re^{i\theta}\colon \theta \in J_r\}$, where J_r is as in Lemma 1. (Note that the definitions of D_1 and D_3 depend on ε.) If for our given ε, $(D_1 \cap D_2)\setminus D_3$ contains an unbounded sequence $r_n e^{i\theta_n}$ we obtain from the definitions of D_1, D_2 and D_3, Lemma 1 and (2.1) that

$$r_n^N < |A(r_n e^{i\theta_n})| < r_n^C + c, \qquad n = 1, 2, \ldots,$$

and this clearly contradicts (3.1) for n large enough.

Thus for arbitrary fixed $\varepsilon > 0$, we may assume that $(D_1 \cap D_2)\setminus D_3$ is bounded. This implies that for $r \geq r_1 \geq r_0$

$$K_r = \{\theta\colon re^{i\theta} \in D_1 \cap D_2\} \subseteq J_r,$$

and thus by Lemma 1 the angular measure of K_r satisfies

$$m(K_r) \leq \varepsilon\pi. \tag{3.4}$$

(We remark that the proof of Theorem 1 would now follow easily from (2.6) and Lemma 3 if D_1 and D_2 were disjoint. As we shall see, (3.4), (2.5) and (2.7) imply that these sets are "essentially" disjoint.)

Define

$$l_1(t) = \begin{cases} 2\pi & \text{if } \theta^*_{D_1}(t) = \infty, \\ \theta^*_{D_1}(t) & \text{otherwise,} \end{cases}$$

$$l_2(t) = \begin{cases} 2\pi & \text{if } \theta^*_{D_2}(t) = \infty, \\ \theta^*_{D_2}(t) & \text{otherwise.} \end{cases}$$

Since D_1 and D_2 are unbounded open sets we have that $l_1(t) > 0$, $l_2(t) > 0$ for t sufficiently large. Also, (3.4) gives $l_1(t) + l_2(t) \leq (2+\varepsilon)\pi$. Now let

$$\overline{\lim_{R\to\infty}} (\log R)^{-1} \pi \int_1^R dt/tl_1(t) = \alpha. \tag{3.5}$$

By definition of l_1, $\alpha \geq 1/2$. Since l_1 and l_2 satisfy the hypotheses of Lemma 3, we obtain

$$\overline{\lim_{R\to\infty}} (\log R)^{-1} \pi \int_1^R dt/tl_2(t) \geq \frac{\alpha}{(2+\varepsilon)\alpha - 1}. \tag{3.6}$$

Define $B_j = \{r\colon \theta^*_{D_j}(r) = \infty\}$, $j = 1, 2$. If $r \in B_1$, $r \geq r_1$, we have $\theta^*_{D_2}(r) \leq \varepsilon\pi$ by (3.4). Thus $B_1 \subseteq \{r\colon \theta^*_{D_2}(r) \leq \varepsilon\pi\}$. By (2.7) we have

$$\overline{\lim_{R\to\infty}} (\log R)^{-1} \int_{B_1\cap[1,R]} dt/t \leq \varepsilon\rho(E). \tag{3.7}$$

Let $\tilde{B}_j = \mathbf{R}^+ \setminus B_j$, $j = 1, 2$. Then (3.3), (2.6) and (3.7) give

$$\begin{aligned} \rho(A) \geq \rho(u) &\geq \overline{\lim_{R\to\infty}} (\log R)^{-1}\pi \int_1^R dt/t\theta^*_{D_1}(t) \\ &= \overline{\lim_{R\to\infty}} (\log R)^{-1}\pi \int_{\tilde{B}_1\cap[1,R]} dt/t\theta^*_{D_1}(t) \\ &= \overline{\lim_{R\to\infty}} (\log R)^{-1} \left[\pi \int_1^R dt/tl_1(t) - \frac{1}{2}\int_{B_1\cap[1,R]} dt/t\right] \\ &\geq \alpha - (\varepsilon/2)\rho(E). \end{aligned} \tag{3.8}$$

Similarly,

$$\overline{\lim_{R\to\infty}} (\log R)^{-1} \int_{B_2\cap[1,R]} dt/t \leq \varepsilon\rho(u) \tag{3.9}$$

and

$$\rho(E) \geq \overline{\lim_{R\to\infty}} (\log R)^{-1} \left[\pi \int_1^R dt/tl_2(t) - \frac{1}{2}\int_{B_2\cap[1,R]} dt/t\right]. \tag{3.10}$$

Thus by (3.9), (3.10) and (3.6) we obtain

$$\rho(E) \geq \frac{\alpha}{(2+\varepsilon)\alpha - 1} - (\varepsilon/2)\rho(u). \tag{3.11}$$

Inequalities (3.8) and (3.11) give

$$\rho(E) \geq \frac{\rho(A) + (\varepsilon/2)\rho(E)}{(2+\varepsilon)(\rho(A) + (\varepsilon/2)\rho(E)) - 1} - (\varepsilon/2)\rho(u).$$

Since ε was arbitrary we obtain

$$\rho(E) \geq \frac{\rho(A)}{2\rho(A) - 1}.$$

This proves (2.4) and hence (1.2). The proof of Theorem 1 is complete.

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