ON THE STABILITY OF ALMOST CONVEX FUNCTIONS

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ABSTRACT. Let $\mathbb{R}$ denote the set of real numbers and $I$ an open interval of $\mathbb{R}$. A function $f: I \to \mathbb{R}$ is said to be almost $\delta$-convex iff $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \delta$ holds for all $(x, y) \in I \times I \setminus N$, where $N \subset I \times I$ is of measure zero, each $t \in [0, 1]$ and some $\delta \geq 0$. It is proved that such a function is uniformly close to a convex function almost everywhere.

Let $\mathbb{R}$ be the set of all real numbers and let $I$ be an open interval (finite or infinite) in $\mathbb{R}$. A function $f: I \to \mathbb{R}$ is called convex iff

\begin{equation}
(f(x) + (1 - t)f(y)
\end{equation}

holds for all pairs $(x, y)$ in $I \times I$ and each $t \in [0, 1]$. A function $f: I \to \mathbb{R}$ is called almost convex iff (1) holds in $I \times I$ except for a set $N \subset I \times I$ of planar Lebesgue measure zero. Let $\delta$ be a nonnegative real number. A function $f: I \to \mathbb{R}$ is called $\delta$-convex iff

\begin{equation}
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \delta
\end{equation}

for all $(x, y) \in I \times I$ and each $t \in [0, 1]$. In case $\delta = 0$, $f: I \to \mathbb{R}$ is the usual convex function. A function $f: I \to \mathbb{R}$ is called almost $\delta$-convex iff (2) holds in $I \times I \setminus N$, where $N \subset I \times I$ is of planar Lebesgue measure zero.

Is every almost convex function equal almost everywhere in $I$ (in the sense of linear Lebesgue measure) to a convex function? Is every almost $\delta$-convex function uniformly close to a convex function almost everywhere? Professor P. M. Gruber raised the former question during a conversation. In what follows we shall answer the latter question in the affirmative and obtain the answer to the former as a corollary to our theorem.

Kuczma [6] proved that an almost mid-convex function, i.e., a function $f: I \to \mathbb{R}$ satisfying

\begin{equation}
f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}
\end{equation}

for all pairs $(x, y)$ in $I \times I \setminus N$, where $N \subset I \times I$ is of planar Lebesgue measure zero, differs from a mid-convex function on a set of linear Lebesgue measure zero.

The question concerning the stability of convex functions has first been raised by Hyers and Ulam [4] and they proved the stability of convex functions. Cholewa [2], in a recent paper, gave a short proof of the fact that a $\delta$-convex function $f: I \to \mathbb{R}$ is uniformly close to a convex function. He also gave an example that mid-convex functions are not stable in the sense of Hyers and Ulam.

N. G. de Bruijn [1] and Jurkat [5], while answering Erdös' problem [3], proved that an almost additive function equals an additive function almost everywhere.

We state and prove a theorem on almost $\delta$-convex functions.

Received by the editors March 25, 1985.
1980 Mathematics Subject Classification. Primary 39C05.
THEOREM. Let \( f: I \to \mathbb{R} \) be a function satisfying

\[
(4) \quad f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \delta
\]

for all pairs \( (x, y) \) in \( I \times I \setminus N \), where \( N \) is a subset of \( I \times I \) of planar Lebesgue measure zero, each \( t \in [0,1] \) and some \( \delta \geq 0 \). Then there exists a convex function \( h: I \to \mathbb{R} \) such that

\[
|f(x) - h(x)| \leq 3\delta \quad \text{for all } x \in I \setminus M,
\]

where \( M \subseteq I \) is of linear Lebesgue measure zero.

PROOF. 1° For a subset \( K \) of \( \mathbb{R} \times \mathbb{R} \) and \( x \in \mathbb{R} \), set

\[
K_x = \{y: (x, y) \in K\} \quad \text{and} \quad T(K) = \{(y, x): (x, y) \in K\}.
\]

Observe that if \( K \) is of planar Lebesgue measure zero, then so is \( T(K) \). Since \( N \cup T(N) \) is of planar measure zero, there therefore exists a subset \( M \subseteq I \) of linear measure zero such that \( x \notin M \) implies \( (N \cup T(N))_x \) is of measure zero. We claim that for \( x, y \in M \) and \( t \in [0,1] \),

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + 2\delta
\]

holds. Let \( z \) be any real number satisfying the inequality \( x < z < y \). Choose \( \varepsilon > 0 \) such that \( \varepsilon < y - z \). Find \( u \notin (N \cup T(N))_y \) and \( v = v(\varepsilon) \notin (N \cup T(N))_x \) satisfying \( x < u < z \) and \( z < y - \varepsilon < v < y \), respectively. Such \( u \) and \( v \) exist since each of \((N \cup T(N))_y \) and \((N \cup T(N))_x \) is of measure zero. It may be noted that \( u \) does not depend on \( \varepsilon \) whereas \( v \) does. We then have

\[
f(z) \leq \frac{v - z}{v - x} f(x) + \frac{z - x}{v - x} f(v) + \delta,
\]

since \( (v, x) \notin N \) and \( x < z < v \), and

\[
f(v) \leq \frac{y - v}{y - v} f(u) + \frac{v - u}{y - u} f(y) + \delta,
\]

since \( (u, y) \notin N \) and \( u < v < y \).

Using (5) and (6), we get

\[
f(z) \leq \frac{v - z}{v - x} f(x) + \frac{z - x}{v - x} \left\{ \frac{y - v}{y - u} f(u) + \frac{v - u}{y - u} f(y) + \delta \right\} + \delta.
\]

On letting \( \varepsilon \to 0 \), \( v \to y \) and (7) becomes

\[
f(z) \leq \frac{y - z}{y - x} f(x) + \frac{z - x}{y - x} f(y) + 2\delta.
\]

2° Set

\[
g(x) = \inf \left( \frac{v - x}{v - u} f(u) + \frac{x - u}{v - u} f(v) \right), \quad x \in I,
\]

where \( \inf \) is taken over \( u, v \notin M \), \( u \neq v \) and satisfying the inequality \( u \leq x \leq v \).

It is immediate from the definition and 1° that

\[
g(x) \geq f(x) - 2\delta \quad \text{for each } x \in I
\]

and

\[
g(x) \leq f(x) \quad \text{for each } x \notin M.
\]
Moreover, for \( x, y \notin M \), \( x < z < y \), on using (8), we have

\[
g(z) \leq \frac{y - z}{y - x} g(x) + \frac{z - x}{y - x} g(y) + 2\delta.
\]

We next show that \( g \) is \( 2\delta \)-convex on \( I \). Let \( u < w < v \) be arbitrary in \( I \) and let \( \varepsilon > 0 \) be given. From the definition of \( g \), it follows that there exist \( a, b \notin M, a \neq b \), \( a \leq u \leq b \) and \( c, d \notin M, c \neq d \), \( c \leq v \leq d \) such that

\[
g(u) + \varepsilon > \frac{b - u}{b - a} f(a) + \frac{u - a}{b - a} f(b)
\]

and

\[
g(v) + \varepsilon > \frac{d - v}{d - c} f(c) + \frac{v - c}{d - c} f(d).
\]

Let \( x, x' \notin M \) satisfy \( a < x < u < w < v < x' < d \). Then

\[
g(w) < \frac{x' - x}{x' - x} g(x) + \frac{w - x}{x' - x} g(x') + 2\delta,
\]

on using (10). Since \( a < x < b, c < x' < d \), on using the definition of \( g \) and (13), we have

\[
g(w) \leq \frac{x' - w}{x' - x} \left( \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b) \right)
\]

\[
+ \frac{w - x}{x' - x} \left( \frac{d - x'}{d - c} f(c) + \frac{x' - c}{d - c} f(d) \right) + 2\delta.
\]

On letting \( x \to u \) and \( x' \to v \) in (14) and using (11) and (12), we get

\[
g(w) \leq \frac{v - w}{v - u} g(u) + \frac{w - u}{v - u} g(v) + 2\delta + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, it follows that \( g: I \to \mathbb{R} \) is \( 2\delta \)-convex.

3° In view of a theorem of Hyers and Ulam [4, Theorem 1], there exists a convex function \( h: I \to \mathbb{R} \) such that

\[
|g(x) - h(x)| \leq \delta \quad \text{for each} \ x \in I.
\]

On using (8) and (9), we obtain

\[
|g(x) - f(x)| \leq 2\delta \quad \text{for each} \ x \in I \setminus M.
\]

(15) and (16) together imply

\[
|f(x) - h(x)| \leq 3\delta \quad \text{for each} \ x \in I \setminus M.
\]

**Corollary.** Let \( f: I \to \mathbb{R} \) be a function satisfying

\[
f(tx + (1 - t)f(y)) \leq tf(x) + (1 - t)f(y)
\]

for all pairs \((x, y) \in I \times I \setminus N\), where \( N \subset I \times I \) is of planar measure zero and each \( t \in [0, 1] \). Then there exists a unique convex function \( h: I \to \mathbb{R} \) such that \( f(x) = h(x) \) for each \( x \in I \setminus M \), where \( M \subset I \) is of linear measure zero.

**Proof.** The existence of \( h \) follows on taking \( \delta = 0 \) in the Theorem above. The facts that a convex function defined on an open interval is continuous and a continuous function which equals zero almost everywhere on an interval is identically zero imply the uniqueness.
REFERENCES


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