

ON THE STABILITY OF ALMOST CONVEX FUNCTIONS

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ABSTRACT. Let \mathbf{R} denote the set of real numbers and I an open interval of \mathbf{R} . A function $f: I \rightarrow \mathbf{R}$ is said to be almost δ -convex iff $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta$ holds for all $(x, y) \in I \times I \setminus N$, where $N \subset I \times I$ is of measure zero, each $t \in [0, 1]$ and some $\delta \geq 0$. It is proved that such a function is uniformly close to a convex function almost everywhere.

Let \mathbf{R} be the set of all real numbers and let I be an open interval (finite or infinite) in \mathbf{R} . A function $f: I \rightarrow \mathbf{R}$ is called convex iff

$$(1) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all pairs (x, y) in $I \times I$ and each $t \in [0, 1]$. A function $f: I \rightarrow \mathbf{R}$ is called almost convex iff (1) holds in $I \times I$ except for a set $N \subset I \times I$ of planar Lebesgue measure zero. Let δ be a nonnegative real number. A function $f: I \rightarrow \mathbf{R}$ is called δ -convex iff

$$(2) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta$$

for all $(x, y) \in I \times I$ and each $t \in [0, 1]$. In case $\delta = 0$, $f: I \rightarrow \mathbf{R}$ is the usual convex function. A function $f: I \rightarrow \mathbf{R}$ is called almost δ -convex iff (2) holds in $I \times I \setminus N$, where $N \subset I \times I$ is of planar Lebesgue measure zero.

Is every almost convex function equal almost everywhere in I (in the sense of linear Lebesgue measure) to a convex function? Is every almost δ -convex function uniformly close to a convex function almost everywhere? Professor P. M. Gruber raised the former question during a conversation. In what follows we shall answer the latter question in the affirmative and obtain the answer to the former as a corollary to our theorem.

Kuczynski [6] proved that an almost mid-convex function, i.e., a function $f: I \rightarrow \mathbf{R}$ satisfying

$$(3) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all pairs (x, y) in $I \times I \setminus N$, where $N \subset I \times I$ is of planar Lebesgue measure zero, differs from a mid-convex function on a set of linear Lebesgue measure zero.

The question concerning the stability of convex functions has first been raised by Hyers and Ulam [4] and they proved the stability of convex functions. Cholewa [2], in a recent paper, gave a short proof of the fact that a δ -convex function $f: I \rightarrow \mathbf{R}$ is uniformly close to a convex function. He also gave an example that mid-convex functions are not stable in the sense of Hyers and Ulam.

N. G. de Bruijn [1] and Jurkat [5], while answering Erdős' problem [3], proved that an almost additive function equals an additive function almost everywhere.

We state and prove a theorem on almost δ -convex functions.

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THEOREM. Let $f: I \rightarrow \mathbf{R}$ be a function satisfying

$$(4) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta$$

for all pairs (x, y) in $I \times I \setminus N$, where N is a subset of $I \times I$ of planar Lebesgue measure zero, each $t \in [0, 1]$ and some $\delta \geq 0$. Then there exists a convex function $h: I \rightarrow \mathbf{R}$ such that

$$|f(x) - h(x)| \leq 3\delta \quad \text{for all } x \in I \setminus M,$$

where $M \subset I$ is of linear Lebesgue measure zero.

PROOF. 1° For a subset K of $\mathbf{R} \times \mathbf{R}$ and $x \in \mathbf{R}$, set

$$K_x = \{y: (x, y) \in K\} \quad \text{and} \quad T(K) = \{(y, x): (x, y) \in K\}.$$

Observe that if K is of planar Lebesgue measure zero, then so is $T(K)$. Since $N \cup T(N)$ is of planar measure zero, there therefore exists a subset $M \subset I$ of linear measure zero such that $x \notin M$ implies $(N \cup T(N))_x$ is of measure zero. We claim that for $x, y \notin M$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + 2\delta$$

holds. Let z be any real number satisfying the inequality $x < z < y$. Choose $\varepsilon > 0$ such that $\varepsilon < y - z$. Find $u \notin (N \cup T(N))_y$ and $v = v(\varepsilon) \notin (N \cup T(N))_x$ satisfying $x < u < z$ and $z < y - \varepsilon < v < y$, respectively. Such u and v exist since each of $(N \cup T(N))_y$ and $(N \cup T(N))_x$ is of measure zero. It may be noted that u does not depend on ε whereas v does. We then have

$$(5) \quad f(z) \leq \frac{v-z}{v-x}f(x) + \frac{z-x}{v-x}f(v) + \delta,$$

since $(v, x) \notin N$ and $x < z < v$, and

$$(6) \quad f(v) \leq \frac{y-v}{y-u}f(u) + \frac{v-u}{y-u}f(y) + \delta,$$

since $(u, y) \notin N$ and $u < v < y$.

Using (5) and (6), we get

$$(7) \quad f(z) \leq \frac{v-z}{v-x}f(x) + \frac{z-x}{v-x} \left\{ \frac{y-v}{y-u}f(u) + \frac{v-u}{y-u}f(y) + \delta \right\} + \delta.$$

On letting $\varepsilon \rightarrow 0$, $v \rightarrow y$ and (7) becomes

$$f(z) \leq \frac{y-z}{y-x}f(x) + \frac{z-x}{y-x}f(y) + 2\delta.$$

2° Set

$$g(x) = \inf \left(\frac{v-x}{v-u}f(u) + \frac{x-u}{v-u}f(v) \right), \quad x \in I,$$

where \inf is taken over $u, v \notin M$, $u \neq v$ and satisfying the inequality $u \leq x \leq v$.

It is immediate from the definition and 1° that

$$(8) \quad g(x) \geq f(x) - 2\delta \quad \text{for each } x \in I$$

and

$$(9) \quad g(x) \leq f(x) \quad \text{for each } x \notin M.$$

Moreover, for $x, y \notin M$, $x < z < y$, on using (8), we have

$$(10) \quad g(z) \leq \frac{y-z}{y-x}g(x) + \frac{z-x}{y-x}g(y) + 2\delta.$$

We next show that g is 2δ -convex on I . Let $u < w < v$ be arbitrary in I and let $\varepsilon > 0$ be given. From the definition of g , it follows that there exist $a, b \notin M$, $a \neq b$, $a \leq u \leq b$ and $c, d \notin M$, $c \neq d$, $c \leq v \leq d$ such that

$$(11) \quad g(u) + \varepsilon > \frac{b-u}{b-a}f(a) + \frac{u-a}{b-a}f(b)$$

and

$$(12) \quad g(v) + \varepsilon > \frac{d-v}{d-c}f(c) + \frac{v-c}{d-c}f(d).$$

Let $x, x' \notin M$ satisfy $a < x < u < w < v < x' < d$. Then

$$(13) \quad g(w) \leq \frac{x'-w}{x'-x}g(x) + \frac{w-x}{x'-x}g(x') + 2\delta,$$

on using (10). Since $a < x < b$, $c < x' < d$, on using the definition of g and (13), we have

$$(14) \quad g(w) \leq \frac{x'-w}{x'-x} \left\{ \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) \right\} + \frac{w-x}{x'-x} \left\{ \frac{d-x'}{d-c}f(c) + \frac{x'-c}{d-c}f(d) \right\} + 2\delta.$$

On letting $x \rightarrow u$ and $x' \rightarrow v$ in (14) and using (11) and (12), we get

$$g(w) \leq \frac{v-w}{v-u}g(u) + \frac{w-u}{v-u}g(v) + 2\delta + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $g: I \rightarrow \mathbf{R}$ is 2δ -convex.

3° In view of a theorem of Hyers and Ulam [4, Theorem 1], there exists a convex function $h: I \rightarrow \mathbf{R}$ such that

$$(15) \quad |g(x) - h(x)| \leq \delta \quad \text{for each } x \in I.$$

On using (8) and (9), we obtain

$$(16) \quad |g(x) - f(x)| \leq 2\delta \quad \text{for each } x \in I \setminus M.$$

(15) and (16) together imply

$$|f(x) - h(x)| \leq 3\delta \quad \text{for each } x \in I \setminus M.$$

COROLLARY. Let $f: I \rightarrow \mathbf{R}$ be a function satisfying

$$f(tx + (1-t)f(y)) \leq tf(x) + (1-t)f(y)$$

for all pairs $(x, y) \in I \times I \setminus N$, where $N \subset I \times I$ is of planar measure zero and each $t \in [0, 1]$. Then there exists a unique convex function $h: I \rightarrow \mathbf{R}$ such that $f(x) = h(x)$ for each $x \in I \setminus M$, where $M \subset I$ is of linear measure zero.

PROOF. The existence of h follows on taking $\delta = 0$ in the Theorem above. The facts that a convex function defined on an open interval is continuous and a continuous function which equals zero almost everywhere on an interval is identically zero imply the uniqueness.

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