SELFADJOINT NONOSCILLATORY SECOND ORDER LINEAR $\mathcal{B}^*$-ALGEBRA DIFFERENTIAL EQUATIONS

MANUEL LOPEZ

ABSTRACT. The main result in this paper states that the second order linear $\mathcal{B}^*$-algebra differential equation $\left(p(t)y\right)'+q(t)y=0$, where $p(t)$ is positive and $q(t)$ is Hermitian for each $t$, is nonoscillatory on $[t_0, \infty)$ if the scalar equation $(\|p^{-1}(t)\|^{-1}W')'+\|q(t)\|W=0$ is nonoscillatory on $[t_0, \infty)$.

Consequently, every criterion on nonoscillation in the scalar case automatically produces another one in the $\mathcal{B}^*$-algebra case.

1. Introduction. There is considerable literature concerning the oscillation of solutions of linear matrix differential equations (see [1, 2, 6, 7] and the references therein). Properties of determinants and the trace are used to obtain some of these results, thereby precluding a straightforward generalization to more general algebras. Hille’s book [3] is devoted to a large extent to generalizing classical results to equations where the dependent variable takes values in a Banach algebra.

In this paper, a method based on an integral inequality of the variational type is used to establish the main result (Theorem 3), which basically says that the oscillatory properties of second order linear $\mathcal{B}^*$-algebra differential equations are closely related to the oscillatory properties of a suitable selfadjoint linear scalar second order differential equation.

Corollaries 1 and 2 show how to apply Theorems 2 and 3, respectively, to obtain nonoscillation criteria for second order linear $\mathcal{B}^*$-algebra differential equations. Corollary 1, in particular, generalizes Reid’s Theorem 5.1 in [6].

Theorem 4 was established in the matrix case by Barrett [1] and Reid [6].

2. Definitions and notation. We shall consider here noncommutative $\mathcal{B}^*$-algebras with an identity $e$ of norm 1. If $B$ is such an algebra, it is well known [9, Theorem 12.41] that $B$ can be identified (up to an isometric $*$-isomorphism) with a closed subalgebra of the algebra of bounded linear operators on some Hilbert space $H$. The symbol $(\cdot, \cdot)$ will denote the inner product of $H$, $\sigma_p(a)$ and $\sigma(a)$ will denote the point spectrum and the spectrum, respectively, of the element $a$ of $B$, and $a^*$ the adjoint of $a$. If $a = a^*$, then $a$ is called Hermitian. If $a$ is Hermitian and $\sigma(a) \subset [0, \infty)$, then $a$ is called positive. The elements of $H$ are called vectors. For other definitions and properties of $\mathcal{B}^*$-algebras see [5 and 9].

All limit processes for functions from $\mathbb{R}$ to $B$ will be considered in the norm topology; primes will denote derivatives with respect to $t$.

Let $L[y] = (p(t)y)' + q(t)y$, $t \in [t_0, \infty)$, where $p(t)$ and $q(t)$ are $B$-valued, $p(t)$ is strictly positive and absolutely continuous and $q(t)$ is Hermitian and Lebesgue integrable on $[t_0, \infty)$. The differential equation

\begin{equation}
L[y] = 0
\end{equation}

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is equivalent to the system
\[(2) \quad y' = p^{-1}(t)z, \quad z' = -q(t)y,\]
and it is well known [3, p. 211] that with the above conditions the initial value problem \(L[y] = 0, y(t_0) = 0, z(t_0) = 0,\) has a solution defined a.e. on \([t_0, \infty).\)

For a function \(y(t)\) we define
\[\{y, y\} = y^*(t)z(t) - z^*(t)y(t),\]
where \(z(t) = p(t)y'(t).\) If the pair \(y, z\) is a solution of (2) such that \(\{y, y\} = 0,\) then \(y, z\) is called a “conjoined” [7, 8] solution of (2).

3. Oscillation theory.

**Definition.** Two points \(s, t\) in \([t_0, \infty)\) are said to be conjugates with respect to (1), or the equivalent system (2), if there exists a nonzero vector solution \(\bar{u}\) of \(L[\bar{u}] = 0\) on \([s, t]\) such that \(\bar{u}(s) = \bar{u}(t) = 0.\)

We say that (1) or (2) is nonoscillatory on an interval if no two distinct points of this interval are conjugate. We say that (1) or (2) is nonoscillatory for large \(t\) if (1) is nonoscillatory on some interval \([c, \infty).\)

The following lemmas are simple extensions to \(B\)-valued functions of similar statements for matrix-valued functions in [7].

**Lemma 1.** If
\[I[\bar{u}; c, d] = \int_c^d [(p(t)\bar{u}', \bar{u}') - (q(t)\bar{u}, \bar{u})] dt > 0\]
for every nonzero vector function \(\bar{u}\) such that \(L[\bar{u}] = 0,\) \(\bar{u}(c) = 0,\) then (1) is nonoscillatory on \([c, d].\)

**Proof.** By integration by parts we have
\[I[\bar{u}; c, d] = (p(t), \bar{u}', \bar{u})|_c^d - \int_c^d (L[\bar{u}], \bar{u}) dt.\]
If (1) is oscillatory, then there exists a nonzero vector solution \(\bar{u}\) of \(L[\bar{u}] = 0\) such that \(\bar{u}(c) = \bar{u}(d) = 0.\) Then since \(I[\bar{u}; c, d] = 0,\) the result follows by contradiction.

**Lemma 2.** Let \(y(t) \in B\) and \(\bar{u}(t) \in H\) for each \(t\) in a given interval. If \(\bar{u}(t) = y(t)\bar{u}(t)\) and \(z(t) = p(t)y',\) then
\[(p(t)\bar{u}', \bar{u}') - (q(t)\bar{u}, \bar{u}) = (p(t)y\bar{u}', y\bar{u}') + (d/dt)(z\bar{u}, \bar{u}) - (\{y, y\}\bar{u}', \bar{u}) - (L[y]\bar{u}, \bar{v}).\]

The proof of Lemma 2 follows from a straightforward computation.

The following theorem relates nonoscillation and invertibility of solutions of \(L[y] = 0.\)

**Theorem 1.** Let \(t_0 \leq c \leq d < \infty.\) Then, for (1) to be nonoscillatory on \([c, d],\) it is necessary that

(i) if \(y_0(t), z_0(t)\) is a solution of (2) satisfying \(L[y_0] = 0,\) \(y_0(c) = 0\) and \(z_0(c)\) is nonsingular, then \(0 \notin \sigma_p(y(t))\) for \(c < t \leq d.\)
It is sufficient that

(ii) there exists a conjoined solution \( y(t), z(t) \) of (2) such that \( 0 \not\in \sigma(y(t)) \) for \( t \in [c, d] \).

**Proof.** The necessity of (i) is obvious. Suppose that (ii) holds. Since \( L[y] = 0 \) and \( \{y, y\} = 0 \), then for an arbitrary nonzero vector \( v(t) \) such that \( v(c) = v(d) = 0 \), we have, by Lemma 2,

\[
\int_c^d \left[ (p(t)v', v') - (q(t)v, v) \right] dt = \int_c^d (p(t)v, v) dt + (z\overline{v}, v)_{d_c}^d
\]

where we have defined \( \overline{u}(t) = y^{-1}(t)v(t) \) for all \( t \in [c, d] \). Since \( p(t) \) is strictly positive, the integral on the right side is positive and the sufficiency of (ii) follows from Lemma 1.

Note. In the matrix case, \( \sigma_p(a) = \sigma(a) \) and Reid [7, Theorem 2.1] shows that (i) and (ii) are equivalent.

The corollary to the following theorem generalizes Reid’s Theorem 5.1 in [6].

**Theorem 2.** If there exist two continuous real-valued functions \( g(t) \) and \( h(t) \) on \([t_0, \infty)\) such that \( p^{-1}(t) \leq g(t)e \), \( q(t) \leq h(t)e \) for all \( t \in [t_0, \infty) \) and the scalar differential equation

\[
(g^{-1}(t)W')' + h(t)W = 0
\]

is nonoscillatory on \([t_0, \infty)\), then (1) is nonoscillatory on \([t_0, \infty)\).

**Proof.** For every nonzero vector solution \( \overline{u} \) of \( L[\overline{u}] = 0 \) we have

\[
I[\overline{u}; c, d] \geq \int_c^d \left( g^{-1}(t)\|\overline{u}'\|^2 - h(t)\|\overline{u}\|^2 \right) dt
\]

where the second inequality follows from the fact that \( \overline{u}(t) \neq 0 \) a.e. Then Schwarz’s inequality gives \( \|\overline{u}'\| \leq \|\overline{u}\| \).

So, by [6, Lemma 5.1] in the scalar case applied to (3), it follows that \( I[\overline{u}; c, d] > 0 \). Hence, (1) is nonoscillatory by Lemma 1.

**Corollary 1.** If there exists a continuous real-valued function \( g(t) \) such that \( p^{-1}(t) \leq g(t)e \), \( q(t) \leq g(t)e \) for all \( t \in [t_0, \infty) \) and \( \int_0^\infty g(t) dt < \infty \), then (1) is nonoscillatory for large \( t \).

**Proof.** The scalar equation \( (g^{-1}(t)W')' + g(t)W = 0 \) has the solution \( W(t) = \sin(-\int_t^\infty g(s) ds) \) which is nonzero for large \( t \). By [6, Lemma 5.1(ii)] it follows that (3) is nonoscillatory for large \( t \), so (1) is nonoscillatory by Theorem 2.

Our main result is

**Theorem 3.** If the scalar equation \( (\|p^{-1}(t)\|^{-1}W')' + \|q(t)\|W = 0 \) is nonoscillatory on \([t_0, \infty)\), then (1) is nonoscillatory on \([t_0, \infty)\).

**Proof.** Define \( g(t) = \|p^{-1}(t)\| \) and \( h(t) = \|q(t)\| \). Then \( p^{-1}(t) \leq g(t)e \) and \( q(t) \leq h(t)e \). Indeed, for each \( t \), the elements \( p^{-1}(t) \) and \( g(t)e \) commute, so we can
embed them in a maximal commutative subalgebra of $B$. If $\mu$ is a multiplicative linear functional, we have

$$\mu(g(t)e - p^{-1}(t)) = \|p^{-1}(t)\| - \mu(p^{-1}(t)) \geq 0.$$ 

So, $p^{-1}(t) \leq g(t)e$ and, similarly, $q(t) \leq h(t)e$.

Thus, the result follows from Theorem 2.

The following corollary is a typical application of Theorem 3.

**COROLLARY 2.** Let $F(t)$ be $B$-valued, continuous and Hermitian on $[t_0, \infty)$. The equation $y'' + F(t)y = 0$ is nonoscillatory for large $t$ if any one of the following conditions is satisfied:

(i) $\limsup_{t \to \infty} t \int_t^\infty \|F(s)\| \, ds < 1/4$;

(ii) $\limsup_{t \to \infty} t^2 \|F(t)\| < 1/4$;

(iii) there exists some positive function $\lambda(t)$ such that

$$\int_0^\infty \Lambda(t) |\lambda''(t) + \|F(t)\| \lambda(t) \, dt < \infty,$$

where $\Lambda(t) = \lambda(t) \int_t^\infty \frac{du}{\lambda^2(u)}$.

**PROOF.** The sufficiency of (i) follows from [4, Corollary 1] and Theorem 3; the sufficiency of (ii) follows from [4, Theorem 9] and Theorem 3 and the sufficiency of (iii) follows from [10] and Theorem 3.

As an application of Corollary 1 we have the following theorem which generalizes theorems of Barrett’s [1, Theorem 3.2] and Reid’s [6, Theorem 5.3].

**THEOREM 4.** If $q(t)$ is strictly positive and continuous on $[t_0, \infty)$ and $\int_0^\infty \|q(t)\| \, dt < \infty$, then the system

$$\begin{align*}
y' &= q(t)z, \\
z' &= -q(t)y
\end{align*}$$

is nonoscillatory for large $t$.

**PROOF.** Taking $g(t) = \|q(t)\|$ for all $t \in [t_0, \infty)$ we have that $q(t) \leq g(t)e$ for all $t \in [t_0, \infty)$. Thus (4) is nonoscillatory by Corollary 1.

**REFERENCES**


