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SELFADJOINT NONOSCILLATORY SECOND ORDER LINEAR B^*-ALGEBRA DIFFERENTIAL EQUATIONS

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ABSTRACT. The main result in this paper states that the second order linear B^*-algebra differential equation $(p(t)y') + q(t)y = 0$, where $p(t)$ is positive and $q(t)$ is Hermitian for each t, is nonoscillatory on $[t_0, \infty)$ if the scalar equation $(\|p^{-1}(t)\|^{-1}W')' + \|q(t)\|W = 0$ is nonoscillatory on $[t_0, \infty)$.

Consequently, every criterion on nonoscillation in the scalar case automatically produces another one in the B^*-algebra case.

1. Introduction. There is considerable literature concerning the oscillation of solutions of linear matrix differential equations (see [**1, 2, 6, 7**] and the references therein). Properties of determinants and the trace are used to obtain some of these results, thereby precluding a straightforward generalization to more general algebras. Hille's book [**3**] is devoted to a large extent to generalizing classical results to equations where the dependent variable takes values in a Banach algebra.

In this paper, a method based on an integral inequality of the variational type is used to establish the main result (Theorem 3), which basically says that the oscillatory properties of second order linear B^*-algebra differential equations are closely related to the oscillatory properties of a suitable selfadjoint linear scalar second order differential equation.

Corollaries 1 and 2 show how to apply Theorems 2 and 3, respectively, to obtain nonoscillation criteria for second order linear B^*-algebra differential equations. Corollary 1, in particular, generalizes Reid's Theorem 5.1 in [**6**].

Theorem 4 was established in the matrix case by Barrett [**1**] and Reid [**6**].

2. Definitions and notation. We shall consider here noncommutative B^*-algebras with an identity e of norm 1. If B is such an algebra, it is well known [**9**, Theorem 12.41] that B can be identified (up to an isometric *-isomorphism) with a closed subalgebra of the algebra of bounded linear operators on some Hilbert space H. The symbol $(\cdot,\cdot)$ will denote the inner product of H, $\sigma_p(a)$ and $\sigma(a)$ will denote the point spectrum and the spectrum, respectively, of the element a of B, and a^* the adjoint of a. If $a = a^*$, then a is called Hermitian. If a is Hermitian and $\sigma(a) \subset [0, \infty)$, then a is called positive. The elements of H are called vectors. For other definitions and properties of B^*-algebras see [**5** and **9**].

All limit processes for functions from $\mathbf{R}$ to B will be considered in the norm topology; primes will denote derivatives with respect to t.

Let $L[y] = (p(t)y')' + g(t)y$, $t \in [t_0, \infty)$, where $p(t)$ and $q(t)$ are B-valued, $p(t)$ is strictly positive and absolutely continuous and $q(t)$ is Hermitian and Lebesgue integrable on $[t_0, \infty)$. The differential equation

$$L[y] = 0 \tag{1}$$

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is equivalent to the system

$$y' = p^{-1}(t)z, \qquad z' = -q(t)y, \tag{2}$$

and it is well known [**3**, p. 211] that with the above conditions the initial value problem $L[y] = 0$, $y(t_0) = 0$, $z(t_0) = 0$, has a solution defined a.e. on $[t_0, \infty)$.

For a function $y(t)$ we define

$$\{y, y\} = y^*(t)z(t) - z^*(t)y(t),$$

where $z(t) = p(t)y'(t)$. If the pair y, z is a solution of (2) such that $\{y, y\} = 0$, then y, z is called a "conjoined" [**7**, **8**] solution of (2).

3. Oscillation theory.

DEFINITION. Two points s, t in $[t_0, \infty)$ are said to be conjugates with respect to (1), or the equivalent system (2), if there exists a nonzero vector solution $\overline{u}$ of $L[\overline{u}] = 0$ on $[s, t]$ such that $\overline{u}(s) = \overline{u}(t) = 0$.

We say that (1) or (2) is nonoscillatory on an interval if no two distinct points of this interval are conjugate. We say that (1) or (2) is nonoscillatory for large t if (1) is nonoscillatory on some interval $[c, \infty)$.

The following lemmas are simple extensions to B-valued functions of similar statements for matrix-valued functions in [**7**].

LEMMA 1. *If*

$$I[\overline{u}; c, d] = \int_c^d [(p(t)\overline{u}', \overline{u}') - (q(t)\overline{u}, \overline{u})]\, dt > 0$$

for every nonzero vector function $\overline{u}$ such that $L[\overline{u}] = 0$, $\overline{u}(c) = 0$, then (1) *is nonoscillatory on $[c, d]$.*

PROOF. By integration by parts we have

$$I[\overline{u}; c, d] = (p(t), \overline{u}', \overline{u})|_c^d - \int_c^d (L[\overline{u}], \overline{u})\, dt.$$

If (1) is oscillatory, then there exists a nonzero vector solution $\overline{u}$ of $L[\overline{u}] = 0$ such that $\overline{u}(c) = \overline{u}(d) = 0$. Then since $I[\overline{u}; c, d] = 0$, the result follows by contradiction.

LEMMA 2. *Let $y(t) \in B$ and $\overline{u}(t) \in H$ for each t in a given interval. If $\overline{v}(t) = y(t)\overline{u}(t)$ and $z(t) = p(t)y'$, then*

$$(p(t)\overline{v}', \overline{v}') - (q(t)\overline{v}, \overline{v}) = (p(t)y\overline{u}', y\overline{u}') + (d/dt)(z\overline{u}, \overline{v}) - (\{y, y\}\overline{u}', \overline{u}) - (L[y]\overline{u}, \overline{v}).$$

The proof of Lemma 2 follows from a straightforward computation.

The following theorem relates nonoscillation and invertibility of solutions of $L[y] = 0$.

THEOREM 1. *Let $t_0 \le c \le d < \infty$. Then, for* (1) *to be nonoscillatory on $[c, d]$, it is necessary that*

(i) *if $y_0(t), z_0(t)$ is a solution of* (2) *satisfying $L[y_0] = 0$, $y_0(c) = 0$ and $z_0(c)$ is nonsingular, then $0 \notin \sigma_p(y(t))$ for $c < t \le d$.*

It is sufficient that

(ii) *there exists a conjoined solution* $y(t), z(t)$ *of* (2) *such that* $0 \notin \sigma(y(t))$ *for* $t \in [c, d]$.

PROOF. The necessity of (i) is obvious. Suppose that (ii) holds. Since $L[y] = 0$ and $\{y, y\} = 0$, then for an arbitrary nonzero vector $\overline{v}(t)$ such that $\overline{v}(c) = \overline{v}(d) = 0$, we have, by Lemma 2,

$$\begin{aligned}\int_c^d [(p(t)\overline{v}', \overline{v}') - (q(t)\overline{v}, \overline{v})]\, dt &= \int_c^d (p(t)\overline{v}, \overline{v})\, dt + (z\overline{u}, \overline{v})|_c^d \\ &= \int_c^d (p(t)y\overline{u}', y\overline{u}')\, dt,\end{aligned}$$

where we have defined $\overline{u}(t) = y^{-1}(t)\overline{v}(t)$ for all $t \in [c, d]$. Since $p(t)$ is strictly positive, the integral on the right side is positive and the sufficiency of (ii) follows from Lemma 1.

Note. In the matrix case, $\sigma_p(a) = \sigma(a)$ and Reid [**7**, Theorem 2.1] shows that (i) and (ii) are equivalent.

The corollary to the following theorem generalizes Reid's Theorem 5.1 in [**6**].

THEOREM 2. *If there exist two continuous real-valued functions* $g(t)$ *and* $h(t)$ *on* $[t_0, \infty)$ *such that* $p^{-1}(t) \leq g(t)e$, $q(t) \leq h(t)e$ *for all* $t \in [t_0, \infty)$ *and the scalar differential equation*

$$(g^{-1}(t)W')' + h(t)W = 0 \tag{3}$$

is nonoscillatory on $[t_0, \infty)$, *then* (1) *is nonoscillatory on* $[t_0, \infty)$.

PROOF. For every nonzero vector solution $\overline{u}$ of $L[\overline{u}] = 0$ we have

$$\begin{aligned}I[\overline{u}; c, d] &\geq \int_c^d (g^{-1}(t)\|\overline{u}'\|^2 - h(t)\|\overline{u}\|^2)\, dt \\ &\geq \int_c^d (g^{-1}(t)\|\overline{u}\|'^2 - h(t)\|\overline{u}\|^2)\, dt,\end{aligned}$$

where the second inequality follows from the fact that $\overline{u}(t) \neq 0$ a.e. Then Schwarz's inequality gives $\|\overline{u}\|' \leq \|\overline{u}'\|$.

So, by [**6**, Lemma 5.1] in the scalar case applied to (3), it follows that $I[\overline{u}; c, d] > 0$. Hence, (1) is nonoscillatory by Lemma 1.

COROLLARY 1. *If there exists a continuous real-valued function* $g(t)$ *such that* $p^{-1}(t) \leq g(t)e$, $q(t) \leq g(t)e$ *for all* $t \in [t_0, \infty)$ *and* $\int^\infty g(t)\, dt < \infty$, *then* (1) *is nonoscillatory for large* t.

PROOF. The scalar equation $(g^{-1}(t)W')' + g(t)W = 0$ has the solution $W(t) = \sin(-\int_t^\infty g(s)\, ds)$ which is nonzero for large t. By [**6**, Lemma 5.1(ii)] it follows that (3) is nonoscillatory for large t, so (1) is nonoscillatory by Theorem 2.

Our main result is

THEOREM 3. *If the scalar equation* $(\|p^{-1}(t)\|^{-1}W')' + \|q(t)\|W = 0$ *is nonoscillatory on* $[t_0, \infty)$, *then* (1) *is nonoscillatory on* $[t_0, \infty)$.

PROOF. Define $g(t) = \|p^{-1}(t)\|$ and $h(t) = \|q(t)\|$. Then $p^{-1}(t) \leq g(t)e$ and $q(t) \leq h(t)e$. Indeed, for each t, the elements $p^{-1}(t)$ and $g(t)e$ commute, so we can embed them in a maximal commutative subalgebra of B. If μ is a multiplicative linear functional, we have

$$\mu(g(t)e - p^{-1}(t)) = \|p^{-1}(t)\| - \mu(p^{-1}(t)) \geq 0.$$

So, $p^{-1}(t) \leq g(t)e$ and, similarly, $q(t) \leq h(t)e$.

Thus, the result follows from Theorem 2.

The following corollary is a typical application of Theorem 3.

COROLLARY 2. *Let $F(t)$ be B-valued, continuous and Hermitian on $[t_0, \infty)$. The equation $y'' + F(t)y = 0$ is nonoscillatory for large t if any one of the following conditions is satisfied:*

(i) $\limsup_{t\to\infty} t \int_t^\infty \|F(s)\|\, ds < 1/4$;

(ii) $\limsup_{t\to\infty} t^2 \|F(t)\| < 1/4$;

(iii) *there exists some positive function $\lambda(t)$ such that*

$$\int^\infty \Lambda(t)|\lambda''(t) + \|F(t)\|\lambda|\, dt < \infty,$$

where $\Lambda(t) = \lambda(t) \int_t^\infty du/\lambda^2(u)$.

PROOF. The sufficiency of (i) follows from [**4**, Corollary 1] and Theorem 3; the sufficiency of (ii) follows from [**4**, Theorem 9] and Theorem 3 and the sufficiency of (iii) follows from [**10**] and Theorem 3.

As an application of Corollary 1 we have the following theorem which generalizes theorems of Barrett's [**1**, Theorem 3.2] and Reid's [**6**, Theorem 5.3].

THEOREM 4. *If $q(t)$ is strictly positive and continuous on $[t_0, \infty)$ and $\int^\infty \|q(t)\|\, dt < \infty$, then the system*

$$y' = q(t)z, \qquad z' = -q(t)y \tag{4}$$

is nonoscillatory for large t.

PROOF. Taking $g(t) = \|q(t)\|$ for all $t \in [t_0, \infty)$ we have that $q(t) \leq g(t)c$ for all $t \in [t_0, \infty)$. Thus (4) is nonoscillatory by Corollary 1.

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