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SIMILARITY-INVARIANT CONTINUOUS FUNCTIONS ON $\mathcal{L}(\mathcal{H})$

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ABSTRACT. Let $f\colon \mathcal{L}(\mathcal{H}) \to X$ be a continuous function from the algebra of all bounded linear operators acting on a complex infinite dimensional Hilbert space $\mathcal{H}$ into a T_1-topological space X. If $f(WAW^{-1}) = f(A)$ for all A in $\mathcal{L}(\mathcal{H})$ and all invertible W, then f is a constant function. The same result is true for a function f satisfying the above conditions defined on a connected open subset of $\mathcal{L}(\mathcal{H})_0 = \{T \in \mathcal{L}(\mathcal{H})\colon\ T \text{ has no normal eigenvalues}\}$.

1. Introduction. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all (bounded linear) operators acting on a complex Hilbert space $\mathcal{H}$. The classical spectral functions (σ (spectrum), sp (spectral radius), σ_e (Calkin essential spectrum), σ_{lre} (Wolf essential spectrum, i.e., the complement in the complex plane $\mathbf{C}$ of the semi-Fredholm domain), etc.), mapping $\mathcal{L}(\mathcal{H})$ (endowed with the norm-topology) into the space X of all compact subsets of $\mathbf{C}$ (Hausdorff metric), or $X =$ the real interval $[0, \infty)$ (the usual topology) have a very erratic behavior.

Indeed, if $\mathcal{H}$ is infinite dimensional, all these functions, and the uncountably many analyzed in [**3**], with the single exception of the spectral radius, are *continuous on a dense subset of* $\mathcal{L}(\mathcal{H})$, and *discontinuous on another dense subset of* $\mathcal{L}(\mathcal{H})$!

The spectral radius, on the other hand, is continuous on an open dense subset of $\mathcal{L}(\mathcal{H})$, *but not everywhere.* (We can also find certain "natural" spectral functions which are discontinuous everywhere; see the above reference.)

Is this behavior a peculiarity of our particular functions? Or, is it possible to construct some "natural" spectral function which is *continuous everywhere*? The answer is: NO. The existence of discontinuities for all these functions is in the nature of things, not a peculiarity of the special functions considered in the literature, and the deep reason is that the spectral functions are *similarity-invariant*; that is, they take the same value on all the elements of the similarity orbit

$$\mathcal{S}(T) = \{WTW^{-1}\colon\ W \in \mathcal{G}(\mathcal{H})\}$$

of an operator T in $\mathcal{L}(\mathcal{H})$. (Here $\mathcal{G}(\mathcal{H})$ denotes the group of all invertible operators.)

THEOREM 1. *Let $f\colon \mathcal{L}(\mathcal{H}) \to X$ be a continuous function defined on $\mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ a complex separable infinite dimensional Hilbert space) with values on a T_1-topological space X.*

Suppose that $f(WTW^{-1}) = f(T)$ for all $T \in \mathcal{L}(\mathcal{H})$, and all $W \in \mathcal{G}(\mathcal{H})$. Then f is a constant function.

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The special case of the spectral radius (continuous on an open dense subset) can be explained as a sort of finite dimensional phenomenon of the following type: Given an operator A, acting on a complex infinite dimensional Banach space $\mathcal{X}$, and $\varepsilon > 0$, there exists $B \in \mathcal{L}(\mathcal{X})$ such that $A-B$ is a finite rank operator, $\|A-B\| < \varepsilon$, and B has a *normal eigenvalue* μ (that is, μ is an isolated point of the spectrum, $\sigma(B)$, of B, and the Riesz spectral subspace $\mathcal{X}(\{\mu\}; B)$ corresponding to the clopen subset $\{\mu\}$ of $\sigma(B)$ is finite dimensional; see [**4**]).

Given B and μ as above, there exist $\eta > 0$ and $\delta > 0$ such that, if $C \in \mathcal{L}(\mathcal{X})$ and $\|B - C\| < \eta$, then $\sigma(C)$ does not intersect the circle $\gamma(\mu; \delta) = \{\lambda \in \mathbf{C}: |\lambda - \mu| = \delta\}$, $\sigma = \sigma(C) \cap \{\lambda: |\lambda - \mu| < \delta\} \neq \emptyset$, and the Riesz spectral subspace $\mathcal{X}(\sigma; C)$ corresponding to the clopen subset σ of $\sigma(C)$ is isomorphic to $\mathcal{X}(\{\mu\}; B)$. For practical purposes, we can directly assume that $\mathcal{X}(\{\mu\}; B)$ and $\mathcal{X}(\sigma; C)$ are identified with $\mathbf{C}^n$ (where $n = \dim \mathcal{X}(\{\mu\}; B)$). Let f be a similarity-invariant function defined on $\mathcal{L}(\mathbf{C}^n)$, with values in a T_1-space X (for instance, $f(T) = \operatorname{trace}(T)$, or $f(T) = \sigma(T)$, or $f(T) = \operatorname{sp}(T)$, etc.); then we can define

$$g: \{C \in \mathcal{L}(\mathcal{X}): \|B - C\| < \eta\} \to X$$

(by using the above identification) via $g(C) = f(C|\mathcal{X}(\sigma; C))$.

It is easily seen that g is continuous, and $g(WCW^{-1}) = g(C)$ for all C in $\mathcal{L}(\mathcal{X})$ and all W in $\mathcal{G}(\mathcal{X})$ such that $\|B - C\| < \eta$, $\|B - WCW^{-1}\| < \eta$.

Let

$$\mathcal{L}(\mathcal{X})_0 = \{T \in \mathcal{L}(\mathcal{X}): \sigma_0(T) = \emptyset\},$$

where $\sigma_0(T)$ denotes the set of all normal eigenvalues of T. It is well known that $\mathcal{L}(\mathcal{X})_0$ is a closed nowhere dense subset of $\mathcal{L}(\mathcal{X})$ [**4**] (see also [**1**, Proposition 11.30]). We have the following result:

PROPOSITION 2. *Let $\mathcal{X}$ be a complex infinite dimensional Banach space. For each B in the open dense subset $\mathcal{L}(\mathcal{X})\backslash\mathcal{L}(\mathcal{X})_0$ (of $\mathcal{L}(\mathcal{X})$), there exists a nonconstant continuous similarity-invariant function g defined on some neighborhood of B.*

(Compare this result and its proof with Proposition 8 of [**3**], on the set of points of continuity of the spectral radius function.)

Similarity-invariant continuous functions from $\mathcal{L}(\mathbf{C}^n)$ into $\mathbf{C}$ admit a very simple characterization. The reader can find a short proof of this characterization in the recently published note of R. Koch [**6**]. (Clearly, $\mathbf{C}$ can be replaced by any T_1-space in the above-mentioned note. The author is deeply indebted to Professor Abraham Sinkov for calling his attention to Koch's article.)

In the case when $\mathcal{H}$ is an infinite dimensional Hilbert space, Proposition 2 is the best possible result on these lines. Indeed, we have

THEOREM 3. *Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space. Let Ω be a connected open (in the relative norm-topology) subset of $\mathcal{L}(\mathcal{H})_0$. If $f: \Omega \to X$ is a continuous function defined on Ω with values on a T_1-space X such that $f(WTW^{-1}) = f(T)$ for all T in $\mathcal{L}(\mathcal{H})_0$ and all W in $\mathcal{G}(\mathcal{H})$ such that T, $WTW^{-1} \in \Omega$, then f is a constant function.*

2. Proof of Theorem 1. We shall need two results on approximation of operators. Recall that $\sigma_e(T) = \sigma(\pi(T))$ ($T \in \mathcal{L}(\mathcal{H})$), where $\pi: \mathcal{L}(\mathcal{H}) \to \mathcal{A}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the canonical projection onto the Calkin algebra. (Here $\mathcal{K}(\mathcal{H})$ denotes the ideal of all compact operators.)

In what follows, $\mathcal{L}(\mathcal{H})_u$ will denote the family of all those operators T in $\mathcal{L}(\mathcal{H})$ with the following property: If μ is an isolated point of $\sigma_e(T)$ and

$$f_{\mu,k}(\lambda) = \begin{cases} (\lambda-\mu)^k & \text{on some neighborhood of } \mu, \\ 0 & \text{on some neighborhood of } \sigma_e(T)\backslash\{\mu\}, \end{cases}$$

then $f_{\mu,k}(\pi(T)) \neq 0$ ($k = 1, 2, \ldots$; $f_{\mu,k}(\pi(T))$ is defined, as usual, via functional calculus).

The first result is a weak version of Theorem 9.1 of [**1**].

PROPOSITION 4. *Let $T \in \mathcal{L}(\mathcal{H})_u$. Then the closure $\mathcal{S}(T)^-$, of $\mathcal{S}(T)$, coincides with the set of all those operators A in $\mathcal{L}(\mathcal{H})$ satisfying the conditions*:

(S)$_0$ $\sigma_0(A) \subset \sigma_0(T)$ *and* $\dim \mathcal{H}(\{\lambda\}; A) = \dim \mathcal{H}(\{\lambda\}; T)$ *for all* λ *in* $\sigma_0(A)$;

(S) *each component of* $\sigma_{lre}(A)$ *intersects* $\sigma_e(T)$;

(F) *the semi-Fredholm domain* $\rho_{\text{s-F}}(A)$ [**2**, **5**], *of A, is a subset of* $\rho_{\text{s-F}}(T)$, *and* $\operatorname{ind}(\lambda - A) = \operatorname{ind}(\lambda - T)$ *for all* $\lambda \in \rho_{\text{s-F}}(A)$;

(F)$_0$ $\min\{\operatorname{nul}(\lambda - A)^k, \operatorname{nul}(\lambda - A)^{*k}\} \geq \min\{\operatorname{nul}(\lambda - T)^k, \operatorname{nul}(\lambda - T)^{*k}\}$ *for all* $\lambda \in \rho_{\text{s-F}}(A)$ *and all* $k = 1, 2, \ldots$.

Furthermore, if

$$\mathcal{S}(\pi(T)) = \{\pi(W)\pi(T)\pi(W)^{-1}\colon\ \pi(W) \in \mathcal{G}[\mathcal{A}(\mathcal{H})]\},$$

then

$$\mathcal{S}(\pi(T))^- = \{\pi(A) \in \mathcal{A}(\mathcal{H})\colon\ A \text{ satisfies (S) and (F)}\}.$$

In particular, $\mathcal{L}(\mathcal{H})_u$ is dense in $\mathcal{L}(\mathcal{H})$, and $\pi[\mathcal{L}(\mathcal{H})_u]$ is dense in $\mathcal{A}(\mathcal{H})$.

The second result is a corollary of the first one, and a standard approximation result (see, e.g., [**2**, Corollary 3.50]).

PROPOSITION 5. *Let $A \in \mathcal{L}(\mathcal{H})$, let $\varepsilon > 0$ and let M be a normal operator such that*

$$\sigma(M) = \{\lambda \in \mathbf{C}\colon\ \operatorname{dist}[\lambda, \sigma_{lre}(A)] \leq \varepsilon/2\}.$$

There exists A' similar to $A \oplus M$ such that $\|A - A'\| < \varepsilon$. Furthermore, if $A \in \mathcal{L}(\mathcal{H})_u$, then $A' \in \mathcal{S}(A)^-$.

Let f and X be as in Theorem 1. Since the singletons are closed in X, and f is continuous and similarity-invariant, we infer that $f|\mathcal{S}(T)^-$ is constant for each T in $\mathcal{L}(\mathcal{H})$.

Let A, $B \in \mathcal{L}(\mathcal{H})_u$. Then Proposition 4 implies that $\mathcal{S}(A)^- \cap \mathcal{S}(B)^- \neq \emptyset$. For instance, the intersection contains every normal operator M such that $\sigma(M) = \{\lambda \in \mathbf{C}\colon |\lambda| \leq \max[\operatorname{sp}(A), \operatorname{sp}(B)]\}$. It follows that $f(A) = f(M) = f(B)$, and therefore f is constant on $\mathcal{L}(\mathcal{H})_u$.

Since, by Proposition 4, $\mathcal{L}(\mathcal{H})_u$ is dense in $\mathcal{L}(\mathcal{H})$, we conclude that f is a constant function. □

REMARK. Theorem 1 remains true (by the same proof) if we merely assume that f is only defined on some ball about $0 \in \mathcal{L}(\mathcal{H})$.

3. Proof of Theorem 3. Let $A \in \Omega$ and let $\varepsilon > 0$ be small enough to guarantee that the intersection $\mathcal{B}(A; \varepsilon)$ of the ball of radius ε about A with $\mathcal{L}(\mathcal{H})_0$ is included in Ω. Let the normal operator M, and let the operator A' similar to $A \oplus M$ be defined as in Proposition 5 (so that $A' \in \mathcal{B}(A; \varepsilon)$).

By Proposition 4, we can find $A'' \in \mathcal{B}(A;\varepsilon)$, $\mathcal{L}(\mathcal{H})_u$ such that $A, A' \in \mathcal{S}(A'')^-$. By using the stability properties of the index and a standard compactness argument, we can find η, $0 < \eta < \varepsilon/2$, such that $\rho_{\text{s-F}}(B) \supset \rho_{\text{s-F}}(A') = \rho_{\text{s-F}}(A)\backslash\sigma(M)$, and $\text{ind}(\lambda - B) = \text{ind}(\lambda - A)$ $(= \text{ind}(\lambda - A'))$ for all $\lambda \in \rho_{\text{s-F}}(A')$ and all B in $\mathcal{L}(\mathcal{H})$ such that $\|A - B\| < \eta$.

We deduce, as in the proof of Theorem 1, that $f(A) = f(A'') = f(A') = f(C)$ for all $C \in \mathcal{B}(A;\eta) \cap \mathcal{L}(\mathcal{H})_u$ such that $\min\{\text{nul}(\lambda - C), \text{nul}(\lambda - C)^*\} = 0$ for all $\lambda \in \rho_{\text{s-F}}(B)$. (Observe that $A' \in \mathcal{S}(C)^{-1}$.)

By Proposition 4, every B in $\mathcal{B}(A;\eta)$ belongs to $\mathcal{S}(C)^-$ for some C as above. But in this case $f(B) = f(C) = f(A)$, so that f is constant on $\mathcal{B}(A;\eta)$.

Since Ω is connected, we conclude that f is constant on Ω. □

Minor modifications of the same argument yield the following analogous result for the Calkin algebra:

THEOREM 6. *A continuous similarity-invariant function h defined on an open connected subset Φ of $\mathcal{A}(\mathcal{H})$, with values on a T_1-space, is necessarily constant on Φ.*

REMARK. Theorems 1, 3, and 6 remain true if $\mathcal{H}$ is nonseparable. Indeed, if $T \in \mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ nonseparable), we can always find a *separable* subspace $\mathcal{R}$ reducing T such that $\sigma(T|\mathcal{R}) = \sigma(T)$, $\sigma_0(T|\mathcal{R}) = \sigma_0(T)$, $\mathcal{H}(\{\lambda\};T|\mathcal{R}) = \mathcal{H}(\{\lambda\};T)$ for all $\lambda \in \sigma_0(T)$, $\sigma_e(T|\mathcal{R}) = \sigma_e(T)$, $\sigma_{lre}(T|\mathcal{R}) = \sigma_{lre}(T)$, and $\text{ind}(\lambda - T|\mathcal{R}) = \text{ind}(\lambda - T)$ for all $\lambda \in \rho_{\text{s-F}}(T)$. Now the results can be proved exactly as in the separable case, by modifying A, B, etc., only on a suitable separable subspace reducing both operators. The details are left to the reader.

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