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UNIFORM DIFFERENTIABILITY, COMPACTNESS, AND l^1

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ABSTRACT. In an earlier paper the authors have shown that conditionally compact subsets of l^1 are characterized by uniform Gâteaux differentiability. Results in this paper show that this equivalence characterizes spaces which contain l^1.

In [**2**] the authors showed that a subset S of an abstract L-space L is relatively weakly compact if and only if there is a point $x \in L$ so that the Gâteaux derivative of the norm at x exists uniformly in the y-direction for all $y \in S$. Since weak convergence and norm convergence of sequences in l^1 coincide, it follows that this uniform differentiability criterion characterizes relative compactness in l^1. In Theorem 1 of this paper, we show that this connection between compactness and differentiability characterizes l^1. Some useful notation and terminology follow.

The letters X, Y, and Z will be reserved for real Banach spaces. The continuous dual of X will be denoted by X^*, and the unit ball of X will be denoted by B_X. Weak and weak* convergence will be denoted by $\xrightarrow{\text{w}}$ and $\xrightarrow{\text{w}^*}$, respectively. If $x, y \in X$, then $D(x,y)(D^+(x,y), D^-(x,y))$ will denote the Gâteaux derivative (one sided derivatives) of the norm at x in the direction y. Since the norm is a convex function, we recall that $D^-(x,y) \leq D^+(x,y)$.

We denote the subgradient of the norm at x by $\partial(x)$, i.e.

$$\partial(x) = \{x^* \in X^*: x^*(y) \leq D^+(x,y), y \in X\}.$$

We note that $\partial: X \to X^*$ is a multivalued mapping which is both monotone and maximal. The reader may consult Rockafellar [**8**, **9**] and Fitzpatrick [**5**] for a discussion of monotone operators. We remark that some of the results in [**5**] require the additional assumption that bounded sequences in X^* have w*-convergent subsequences. Since the spaces X in which we wish to apply Fitzpatrick's results are separable, this added condition is satisfied automatically in our setting.

We say that uniform Gâteaux differentiability (= u.G.d.) characterizes compactness in X provided that the norm closure of a subset K of X is compact iff there is some $x \in X$ so that $D(x,y)$ exists uniformly for $y \in K$. We remark that Examples 2.6 and 2.7 of [**2**] show that u.G.d. does not characterize even weak compactness in general.

Now we come to the main result of our paper. Our proof makes use of Day's locally uniformly convex norm γ on c_0 and renorming techniques of Troyanski [**11**]. We note that γ enjoys the following monotonicity property: If $(a_n), (b_n) \in c_0$ and $|a_n| \leq |b_n|$ for n, then $\gamma((a_n)) \leq \gamma((b_n))$. The reader may consult Day [**3**], Diestel [**4**, pp. 94–100], or Rainwater [**7**] for properties of γ.

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THEOREM 1. *An infinite dimensional Banach space X contains an isomorphic copy of l^1 iff there is an infinite dimensional subspace Y of X and an equivalent norm $|||\cdot|||$ on Y so that uniform Gâteaux differentiability characterizes compactness in $(Y, |||\cdot|||)$.*

The following fact plays a pivotal role in establishing Theorem 1. Because of its importance in the proof, we have chosen to designate it as a lemma and to proceed with its proof before presenting our argument for Theorem 1.

LEMMA 2. *If X is a Banach space with a weakly Cauchy normalized basis (x_n), then there is a dense G_δ-subset A of X so that $D(x, x_n)$ exists uniformly in n for all $x \in A$.*

PROOF. Suppose that (x_n) satisfies the hypotheses, and let $||\cdot||$ denote the given norm on X. Let $(e_n)_{n=1}^{\infty}$ be the usual basis for c_0, and let $e_0 = (1, 1, 1, \ldots)$. Let $u \in c$, and set $\tau_n(u) = \operatorname{dist}(u, [e_0, \ldots, e_n])$, $n = 0, 1, 2, \ldots$, where $[e_0, \ldots, e_n]$ denotes the closed linear span of $\{e_0, \ldots, e_n\}$ in c and the distance is calculated in c.

Now suppose that $x^* \in X^*$ and define $L: X^* \to c$ by

$$L(x^*) = (x^*(x_1), x^*(x_2), \ldots).$$

Let $T: X^* \to c_0$ be the nonlinear operator defined by

$$T(x^*) = (||x^*||, x^*(x_1), \tau_0(L(x^*)),\ x^*(x_2 - x_1), \tau_1(L(x^*)),\\ x^*(x_3 - x_2),\ \tau_2(L(x^*)), \ldots).$$

Since Day's norm γ satisfies the monotonicity condition cited before Theorem 1, T is injective, and $\gamma \circ T$ is strictly positive definite and absolutely homogeneous, it follows that $\gamma \circ T$ defines a norm on X^*. Further, since γ is an equivalent norm on c_0 and $||x^*||$ is the first component of $T(x^*)$, it follows that $\gamma \circ T$ is an equivalent norm on X^*. Let $|||\cdot|||$ denote this norm on X^*.

We assert that $|||\cdot|||$ is a dual norm on X^*. To demonstrate this, it suffices to show that $B_{(X^*, |||\cdot|||)}$ is w*-closed. Suppose then that $(x_\alpha^*)_{\alpha \in \Delta}$ is a net so that $|||x_\alpha^*||| \leq 1$ for each α, and $x_\alpha^* \xrightarrow{w^*} x^*$. Clearly $||x^*|| \leq \underline{\lim}||x_\alpha^*||$ since $||\cdot||$ is a dual norm. Further, $x^*(x_1) = \lim_\alpha x_\alpha^*(x_1)$ and $x^*(x_i - x_{i-1}) = \lim_\alpha x_\alpha^*(x_i - x_{i-1})$. Next we consider $\tau_n(L(x^*))$. Fix n and suppose that $u_\alpha \in [e_0, \ldots, e_n]$ so that $||L(x_\alpha^*) - u_\alpha||_\infty = \tau_n(L(x_\alpha^*)), \alpha \in \Delta$. Since $\{x_\alpha^* : \alpha \in \Delta\}$ is norm bounded and L is a bounded linear operator, $\{u_\alpha : \alpha \in \Delta\}$ must be a bounded subset of the finite dimensional space $[e_0, \ldots, e_n]$. Let u be a norm cluster point of (u_α); without loss of generality, suppose that $||u_\alpha - u||_\infty \to 0$. Now $L(x_\alpha^*) \xrightarrow{w^*} L(x^*)$ in l^∞, and thus $L(x_\alpha^*) - u_\alpha \xrightarrow{w^*} L(x^*) - u$ in l^∞. Therefore

$$||L(x^*) - u||_\infty \leq \underline{\lim}||L(x_\alpha^*) - u_\alpha||,$$

i.e. $\tau_n(L(x^*)) \leq \underline{\lim}\tau_n(L(x_\alpha^*))$. Thus each "coordinate" of T is w*-lower semicontinuous, and it follows that $|||x^*||| = \gamma(T(x^*)) \leq 1$. Hence $|||\cdot|||$ is an equivalent dual norm on X^*.

Next we claim that $|||\cdot|||$ is a strictly convex norm on X^*. Suppose that $|||x^*||| = |||y^*||| = |||(x^*+y^*)/2|||$. By the monotonicity of γ, we have that

$$|||x^*||| = \frac{\gamma(T(x^*+y^*))}{2} \le \frac{\gamma(T(x^*)+T(y^*))}{2} \le \frac{\gamma(T(x^*))}{2} + \frac{\gamma(T(y^*))}{2} = |||x^*|||.$$

The local uniform convexity of γ then certainly guarantees that $\gamma(T(x^*)-T(y^*)) = 0$, i.e. $T(x^*) = T(y^*)$. Since T is injective, it follows that $x^* = y^*$, and indeed $|||\cdot|||$ is a strictly convex norm on X^*. Of course, we note that the strict convexity of $|||\cdot|||$ on X^* guarantees that the induced norm on X is smooth (e.g. see Diestel [**4**, p. 23]).

Now let ∂ denote the maximal monotone operator in $X \times X^*$ given by the subgradient of the original norm $||\cdot||$ on X, i.e.

$$\begin{aligned}\partial(x) &= \{x^* \in X^* : x^*(y) \le D^+(x,y) \text{ for all } y \in X\}\\ &= \{x^* \in X^* : D^-(x,y) \le x^*(y) \le D^+(x,y) \text{ for all } y \in X\}.\end{aligned}$$

We note that if $x \neq 0$, then $\partial(x) = \{x^* \in X^* : ||x^*|| = 1 \text{ and } x^*(x) = ||x||\}$. Next set $\theta(x) = \inf\{|||x^*||| : x^* \in \partial(x)\}$. Then by Lemma 2 and Corollary 3 of Fitzpatrick [**5**], θ is continuous on a dense G_δ-subset A of X. Further, by Lemma 4 of [**5**], $|||x^*||| = \theta(x)$ for each $x^* \in \partial(x)$ whenever $x \in A$. Since $\partial(x)$ is a convex set and $|||\cdot|||$ is strictly convex on X^*, it follows that $\partial(x)$ is a singleton whenever $x \in A$. Thus if $x \in A$, then there is a unique $x^* \in B_{X^*}$ so that $x^*(x) = ||x||$, i.e. the original norm on X is smooth at x. Consequently, if $x \in A$ and $x^* = \partial(x)$, then $D(x,y) = x^*(y)$ for $y \in X$.

In addition we note that if $x_0 \in A$ and (x_α, x_α^*) is a net from ∂ so that $||x_\alpha - x_0|| \to 0$ and $x_\alpha^* \xrightarrow{w^*} x_0^*$, then the maximality of ∂ ensures that $(x_0, x_0^*) \in \partial$. That is, $\langle x_0^* - x^*, x_0 - x\rangle \ge 0$ for each $(x, x^*) \in \partial$, and, since $\partial(x_0)$ is a singleton and ∂ is a maximal, $x_0^* = \partial(x_0)$. Now suppose that $||y_n - x_0|| \to 0$ and $y_n^* \in \partial(y_n)$ for each n. Let y^* be a w*-cluster point of (y_n^*) (recall that (y_n^*) is norm bounded), and let $(y_{n_\alpha}^*)$ be a subnet of (y_n^*) so that $y_{n_\alpha}^* \xrightarrow{w^*} y^*$. But then from the preceding observation in this paragraph, it follows that $y^* = x_0^*$. Therefore $y_n^* \xrightarrow{w^*} x_0^*$, and ∂ is (norm,w*)-continuous at each point of A.

Suppose then that $x_0 \in A$, $x_0 \neq \theta$, and $x_0^* = \partial(x_0)$. We assert that

$$\lim_{t\to 0} \frac{||x_0 + tx_n|| - ||x_0||}{t} = x_0^*(x_n)$$

uniformly in n. Suppose not. Let $\varepsilon > 0$, let (t_i) be a null sequence, and let (n_i) be a sequence of positive integers so that

$$\left|\frac{||x_0 + t_i x_{n_i}|| - ||x_0||}{t_i} - x_0^*(x_{n_i})\right| > \varepsilon$$

for each i. By passing to a subsequence and multiplying the basis elements by -1 if necessary, we may (and shall) assume that $t_i > 0$ for each i. Since the difference quotients of convex functions are monotone, we see that

$$(\#) \qquad ||x_0 + t_i x_{n_i}|| > t_i(x_0^*(x_{n_i}) + \varepsilon) + ||x_0||$$

for each i. Now choose $u_{n_i}^* \in \partial(x_0 + t_i x_{n_i})$ so that

$$|||u_{n_i}^*||| < \theta(x_0 + t_i x_{n_i}) + 1/i, \qquad i = 1, 2, \ldots.$$

Since θ is continuous at x_0, it follows that $\theta(x_0 + t_i x_{n_i}) \to \theta(x_0) = |||x_0^*|||$. Hence $|||u_{n_i}^*||| \to |||x_0^*|||$. And since $x_0 + t_i x_{n_i} \to x_0$, the (norm, w*)-continuity of ∂ at x_0 ensures that $u_{n_i}^* \xrightarrow{w^*} x_0^*$. Then by the local uniform convexity and monotonicity of γ, we see that $\gamma[T(u_{n_i}^*) - T(x_0^*)] \to 0$. But γ is an equivalent norm on c_0; thus $||T(u_{n_i}^*) - T(x_0^*)||_\infty \to 0$. Let $k \in \mathbf{N}$ so that $||T(u_{n_i}^*) - T(x_0^*)||_\infty < \varepsilon/8$ for $i \geq k$. Therefore

$$|u_{n_i}^*(x_1) - x_0^*(x_1)| \leq \varepsilon/8 \quad \text{for } i \geq k,$$
$$|\langle u_{n_i}^* - x_0^*, x_p - x_{p-1}\rangle| < \varepsilon/8 \quad \text{for } i \geq k \text{ and } p \geq 2,$$

and

$$|\tau_j(L(u_{n_i}^*)) - \tau_j(L(x_0^*))| < \varepsilon/8 \quad \text{for } i \geq k \text{ and all } j.$$

Let $j_0 \in \mathbf{N}$ so that $\tau_j(L(x_0^*)) < \varepsilon/8$ for $j \geq j_0$. Therefore $\tau_j(L(u_{n_i}^*)) < \varepsilon/4$ for $i \geq k$ and $j \geq j_0$. But this says that $\{L(u_{n_i}^*): i \in \mathbf{N}\}$ must be totally bounded in c (or l^∞). Without loss of generality, suppose that $L(u_{n_i}^*) \to u$ in c. But then certainly $(L(u_{n_i}^*))_{i=1}^\infty$ converges coordinatewise to u. However, $(L(u_{n_i}^*))$ converges coordinatewise to $L(x_0^*)$ since $(L(u_{n_i}^*))_{i=1}^\infty \xrightarrow{w^*} L(x_0^*)$ in l^∞. Thus, $u = L(x_0^*)$, i.e. if $|||z_i^*||| \to |||z^*|||$ and $z_i^* \xrightarrow{w^*} z^*$, then $L(z_i^*) \to L(z^*)$ in l^∞. Now choose n_i so that $|\langle u_{n_i}^* - x_0^*, x_{n_i}\rangle| < \varepsilon/2$ and $u_{n_i}^*(x_{n_i}) < x_0^*(x_{n_i}) + \varepsilon/2$. Therefore

$$\begin{aligned} ||x_0 + t_i x_{n_i}|| &= u_{n_i}^*(x_0 + t_i x_{n_i}) = u_{n_i}^*(x_0) + t_i u_{n_i}^*(x_{n_i}) \\ &\leq ||x_0|| + t_i(x_0^*(x_{n_i}) + \varepsilon/2), \end{aligned}$$

and we have a direct contradiction of (#). The lemma follows.

PROOF OF THEOREM 1. Suppose the Banach space X contains an isomorphic copy of l^1. Let $|||\cdot|||$ denote the usual norm on l^1. Then, by Theorem 2.5 of Bilyeu and Lewis [**2**], u.G.d. characterizes weak compactness in $(l^1, |||\cdot|||)$. Since weakly convergent sequences in l^1 are norm convergent, it follows that u.G.d. characterizes compactness in l^1.

Now suppose that l^1 does not embed in X, and let Y be any infinite dimensional closed linear subspace of X. Let $||\cdot||$ be an equivalent norm on Y. Then by a classical result of Bessaga and Pełczynski [**1**], $(Y, ||\cdot||)$ contains an infinite dimensional subspace with a normalized basis (x_n). By Rosenthal's fundamental characterization of spaces containing l^1 [**10**], (x_n) has a weak Cauchy subsequence. Since any subsequence of a basis is a basic sequence [**6**, p. 6], we may (and shall) assume that Y has a normalized weakly Cauchy basis (x_n). But then by Lemma 2 there is a point $x \in Y$ so that $D(x, x_n)$ exists uniformly for $n \in \mathbf{N}$. But (x_n) cannot have a subsequence which is norm convergent. For if $x_{n_i} \to x$, then $x_k^*(x) = \lim_i x_k^*(x_{n_i}) = 0$ for each coefficient functional x_k^*, and $x = 0$. But $||x|| = \lim_i ||x_{n_i}|| = 1$. Therefore u.G.d. does not characterize compactness in $(Y, ||\cdot||)$, and the theorem follows.

A companion question that one might ask in view of Theorem 1 is whether u.G.d. characterizes compactness in every equivalent norm on l^1. The following result shows that this is definitely not the case.

THEOREM 3. *Let X be a Banach space with a Schauder basis $(x_n)_{n=1}^{\infty}$, and let x_0 be any nonzero point in X. Then there is an equivalent norm $|||\cdot|||$ on X so that $|||\cdot|||$ is Fréchet differentiable at x_0.*

PROOF. Let X be a Banach space with basis (y_n), and let (α_n) be a sequence of nonzero scalars so that $(\alpha_n y_n)$ is weakly Cauchy. Let $x_n = \alpha_n y_n$, $n \in \mathbf{N}$, and let $x_0 \in X$, $x_0 \neq 0$. Further, suppose that (L_n) is a sequence of functionals defined on X^* as follows:

$$L_n(x^*) = x^*(x_n - x_{n-1}), \quad n \geq 2, \qquad L_1(x^*) = x^*(x_1).$$

Now define $T: X^* \to c_0$ by

$$T(x^*) = (||x^*||, L_1(x^*), L_2(x^*), L_3(x^*), \ldots),$$

and define ρ^* on X^* by $\rho^*(x^*) = \gamma(T(x^*))$. Then, as in the proof of Lemma 2, it follows that ρ^* is an equivalent strictly convex dual norm on X^*. Let ρ be the equivalent induced smooth norm on X. Now let x_0^* be the unique ρ^*-norm-one member of X^* so that $x_0^*(x_0) = \rho(x_0)$. Let $|||\cdot|||$ be defined on X^* by

$$|||x^*||| = \gamma(\tau_0(x^*); T(x^*)) = \gamma(\tau_0(x^*), ||x^*||, L_1(x^*), \ldots),$$

where $\tau_0(x^*) = \rho^* - \operatorname{dist}(x^*, [x_0^*])$. By our earlier argument in the proof of Lemma 2, we see that $|||\cdot|||$ is an equivalent strictly convex dual norm on X^* with the property that $|||y_i^* - x_0^*||| \to 0$ whenever $|||y_i^*||| \to |||x_0^*|||$ and $y_i^* \xrightarrow{w^*} x_0^*$. We denote the induced norm on X by $|||\cdot|||$ also.

Next we make some observations about the norms of the specific elements x_0 and x_0^*:

(a) $|||x_0^*||| = \rho^*(x_0^*)$ since $\tau_0(x_0^*) = 0$,

(b) $|||x^*||| \geq \rho^*(x^*)$ for all $x^* \in X^*$ because of the monotonicity of γ, and

(c) $|||x||| \leq \rho(x)$ because of the inequality in (b).

Therefore $|||x_0||| \leq \rho(x_0)$. But $|||x_0^*||| = \rho^*(x_0^*) = 1$, $x_0^*(x_0) = \rho(x_0)$, and thus $|||x_0||| = \rho(x_0)$.

Now let ∂ denote the subgradient of $|||\cdot|||$ in $X \times X^*$. Suppose that $|||y_i - x_0||| \to 0$. By the maximality of ∂ and the metrizability of $(B_{(X^*, |||\cdot|||)}, \mathrm{w}^*)$, we may assume that $\partial(y_i) \xrightarrow{w^*} \partial x_0 = x_0^*$. But $|||\partial(y_i)||| = |||\partial x_0||| = 1$ for each i. Therefore $|||\partial y_i - \partial x_0||| \to 0$. Thus ∂ is continuous at x_0, and $|||\cdot|||$ is Fréchet differentiable at those points where ∂ is continuous [**4**, p. 30]. The theorem follows.

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