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ON WEAKLY COMPACT OPERATORS ON SPACES OF VECTOR VALUED CONTINUOUS FUNCTIONS[1]

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ABSTRACT. Let K and S be compact Hausdorff spaces and θ a continuous function from K onto S. Then for any Banach space E the map $f \mapsto f \circ \theta$ isometrically embeds $C(S, E)$ as a closed subspace of $C(K, E)$. In this note we prove that when E' has the Radon-Nikodým property, every weakly compact operator on $C(S, E)$ can be lifted to a weakly compact operator on $C(K, E)$. As a consequence, we prove that the compact dispersed spaces K are characterized by the fact that $C(K, E)$ has the Dunford-Pettis property whenever E has.

For a Banach space E and a compact Hausdorff space K, $C(K, E)$ will denote the Banach space of all E-valued continuous functions on K under the supremum norm. We shall identify, as usual, the topological dual of $C(K, E)$ with the space $\text{rcabv}(\text{Bo}(K), E')$ of the regular, countably additive Borel measures on K of bounded variation, with values in the topological dual E' of E, endowed with the variation norm [**3**, Theorem 2.2].

The notation and terminology used and not defined can be found in [**6**, **10**, or **11**].

Let K and S be compact Hausdorff spaces, and let θ: $K \to S$ be an onto continuous function. For any Banach space E, the map $f \mapsto J_\theta(f) = f \circ \theta$ is then a linear isometry which embeds $C(S, E)$ as a closed subspace of $C(K, E)$. The following result establishes that, in some cases, every weakly compact operator on $C(S, E)$ can be lifted to a weakly compact operator on $C(K, E)$.

THEOREM 1. *Let K, S and θ be as above, and let E be a Banach space such that E' has the Radon-Nikodým property. Then, if F is a Banach space and U: $C(S, E) \to F$ is a weakly compact operator, there exists a weakly compact operator $\overline{U}$: $C(K, E) \to F$ so that $\|\overline{U}\| = \|U\|$ and $\overline{U} \circ J_\theta = U$.*

PROOF. Let D be the closed unit ball of F'. Then $U'(D)$ is weakly compact in $\text{rcabv}(\text{Bo}(S), E')$, and so there is a control measure λ for $U'(D)$ [**3**, Proposition 3.1; **6**, I.2.4]; i.e., a positive Radon measure on S such that $\lim_{\lambda(B) \to 0} \|m(B)\| = 0$, uniformly in $m \in U'(D)$. Because E' has the Radon-Nikodým property, for each $x' \in D$ there exists a Bochner integrable density $g_{x'} \in L^1(S, \lambda, E')$ for $U'(x')$. Recalling that the map $g \mapsto m_g$, where $m_g(f) = \int \langle f, g \rangle \, d\lambda$ for each f in $C(S, E)$, is an isometry from $L^1(S, \lambda, E')$ into $\text{rcabv}(\text{Bo}(S), E')$, it follows that $\{ g_{x'}: x' \in D \}$ is weakly compact in $L^1(S, \lambda, E')$.

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Now let μ be a positive Radon measure on K such that $\theta(\mu) = \lambda$ [**11**, Chapter I, Theorem 12]. The map

$$L^1(S, \lambda, E') \ni g \mapsto g \circ \theta \in L^1(K, \mu, E')$$

is a linear isometry, and so $\{g_{x'} \circ \theta\colon\ x' \in D\}$ is a weakly compact subset of $L^1(K, \mu, E')$.

For $f \in C(K, E)$ and $x' \in F'$ let us define

$$\langle \overline{U}(f), x' \rangle = \int \langle f, g_{x'} \circ \theta \rangle \, d\mu.$$

$\overline{U}(f)$ clearly belongs to the algebraic dual of F'. But

$$|\langle \overline{U}(f), x' \rangle| \leqslant \|f\|_\infty \int \|g_{x'} \circ \theta\| \, d\mu = \|f\|_\infty \|U'(x')\|$$
$$\leqslant \|U\| \, \|f\|_\infty \|x'\|,$$

which proves that $\overline{U}(f) \in F''$, and so $\overline{U}$ is a continuous linear operator from $C(K, E)$ to F'' with $\|\overline{U}\| \leqslant \|U\|$. Besides, $\overline{U} \circ J_\theta = U$, and then $\|\overline{U}\| = \|U\|$.

Let $(x_i')_{i \in I} \subset D$ be a weak* null net. Then $(U'(x_i'))_{i \in I}$ is relatively weakly compact and weak* convergent to 0. It follows that it is weakly null, and so $(g_{x_i'} \circ \theta)_{i \in I}$ converges weakly to 0 in $L^1(K, \mu, E')$. Therefore,

$$\lim_{i \in I} \langle \overline{U}(f), x_i' \rangle = 0.$$

This shows that the restriction of $\overline{U}(f)$ to D is weak* continuous, and so, by Grothendieck's theorem [**9**, 3.11.4], $\overline{U}(f)$ belongs to F for every f in $C(K, E)$.

Finally, let us note that $\overline{U}'(D) = \{g_{x'} \circ \theta\colon\ x' \in D\}$ (with the canonical identification of $L^1(K, \mu, E')$ as a closed subspace of rcabv(Bo(K), E')), and so $\overline{U}'$, and consequently $\overline{U}$, is weakly compact.

COROLLARY. *Under the assumptions of Theorem* 1, *if* (x_n') *is a weakly convergent sequence in* rcabv(Bo(S), E'), *there exists a weakly convergent sequence* (y_n') *in* rcabv(Bo(K), E') *such that* $J_\theta'(y_n') = x_n'$ *for every* n.

PROOF. We can assume that (x_n') is weakly null. The operator $U\colon C(S, E) \to c_0$ defined by $U(f) = (\langle f, x_n' \rangle)$ is then weakly compact. According to Theorem 1, there is a weakly compact operator $\overline{U}\colon C(K, E) \to c_0$ such that $\overline{U} \circ J_\theta = U$. If $\overline{U}(f) = (\langle f, y_n' \rangle)$ for f in $C(K, E)$, the sequence (y_n') fulfills the requirements of the corollary.

Recall that a Banach space E is said to have the Dunford-Pettis property (D.P.P.) if every weakly compact operator on E sends weakly compact sets into norm compact ones. This property was introduced by A. Grothendieck in his important paper [**8**] and has been intensively studied (see [**5**]). The long-standing open question of whether $C(K, E)$ has the D.P.P. if E has, was answered by M. Talagrand [**12**], who built a Banach space T such that

(1) T and T' have unconditional basis.

(2) T' is a Schur space; in particular, T and T' have the D.P.P.

(3) $C([0, 1], T)$ does not have the D.P.P.

However, if K is a dispersed compact space (i.e., a space which does not contain any perfect set; see [**10**, §5]), it is known that $C(K, E)$ has the D.P.P. if E has (see [**4** and **7**]). Now we can prove that this property indeed characterizes the compact dispersed spaces.

THEOREM 2. *Let K be a compact Hausdorff space. The following properties are equivalent*:

(a) *K is dispersed.*

(b) *If E is a Banach space with the D.P.P., so is $C(K, E)$.*

(c) *If T is the Talagrand space, $C(K, T)$ has the D.P.P.*

PROOF. That (a) implies (b) is known (see [**4**, Theorem 4; **7**, Theorem 13]) and (b) $\Rightarrow$ (c) is obvious. Let us suppose that K is not dispersed. Then there is a continuous onto function θ: $K \to [0,1]$ [**10**, §2.4.2]. According to Talagrand's result, $C([0,1], T)$ does not have the D.P.P., and so there is a Banach space F, a weakly compact operator U from $C([0,1], T)$ to F and a weakly compact subset H of $C([0,1], T)$ such that $U(H)$ is not norm compact. Because T' is separable, it has the Radon-Nikodým property [**6**, III.3.1]. By Theorem 1, there is a weakly compact operator $\overline{U}$ from $C(K, T)$ into F such that $\overline{U} \circ J_\theta = U$. But then $J_\theta(H)$ is weakly compact and $\overline{U}(J_\theta(H)) = U(H)$ is not norm compact. This shows that $C(K, T)$ does not have the D.P.P., and concludes the proof.

ADDITIONAL REMARK. After the writing of this paper, some other extension theorems for operators on $C(S, E)$ have been obtained in [**2**]. By using the methods of this last paper (essentially the choice of a weak* density instead of a Bochner one), one can prove that Theorem 1 is true without assuming the Radon-Nikodým property on E'. For details, we refer to [**2**].

We should also mention that other characterizations of compact dispersed spaces in terms of operators on spaces of continuous vector valued functions can be seen in [**1**].

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