COMPOSITION OF LINEAR FRACTIONAL TRANSFORMATIONS IN TERMS OF TAIL SEQUENCES

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ABSTRACT. We consider sequences \( \{s_n\} \) of linear fractional transformations. Connected to such a sequence is another sequence \( \{S_n\} \) of linear fractional transformations given by

\[
S_n = s_1 \circ s_2 \circ \cdots \circ s_n, \quad n = 1, 2, 3, \ldots.
\]

We introduce a new way of representing \( s_n \) (in terms of so-called tail sequences). This representation is established to give nice expressions for \( S_n \). It can be seen as a generalization of the canonical form for \( s_n \), which gives nice expressions for

\[
T_k = s_n \circ s_{n-1} \circ \cdots \circ s_{n-k}, \quad (k \text{ terms})
\]

1. Introduction. A linear fractional transformation

\[
s(w) = \frac{a + cw}{b + dw}, \quad a, b, c, d \in \mathbb{C}, \Delta = ad - bc \neq 0,
\]

is a meromorphic function which maps the extended complex plane \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) bijectively onto itself. Hence, we adopt the convention that

\[
a + cw = a \quad \text{if } c = 0 \text{ and } w = \infty,
\]

and

\[
b + dw = b \quad \text{if } d = 0 \text{ and } w = \infty.
\]

(It should be noted that the notation (1.1) used in this paper differs from the standard notation \((aw + b)/(cw + d)\).) If \( s(w) \neq w \), then \( s \) has two fixed points, \( x \) and \( y \), which coincide iff \( s \) is parabolic. It is well known that then

\[
\frac{s(w) - x}{s(w) - y} = \frac{b + dy}{b + dx} \cdot \frac{w - x}{w - y} \quad \text{if } x \neq y,
\]

and

\[
\frac{1}{s(w) - y} = \frac{2d}{b + c} + \frac{1}{w - y} \quad \text{if } x = y.
\]

(We define

\[
w - y = 0 \quad \text{if } w = y = \infty, \quad w - x = 0 \quad \text{if } w = x = \infty.
\]

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Clearly, \( x \) and \( y \) are both finite iff \( d \neq 0 \). Hence, if \( d \neq 0 \), then (1.4)–(1.5) is an alternative way to write (1.1). The following example illustrates one of the advantages obtained by this way of representing \( s \).

**Example 1.1.** Given the linear fractional transformation (1.1) with \( d \neq 0 \) and fixed points \( x \) and \( y \) (\( s(w) \neq w \) since \( d \neq 0 \)), the composition of \( n \) such transformations,

\[
S_n(w) = s \circ s \circ \cdots \circ s(w), \quad n \in \mathbb{N},
\]
is also a linear fractional transformation. If \( s \) is nonparabolic, then \( x \neq y \), and by (1.4)

\[
\frac{S_n(w) - x}{S_n(w) - y} = \frac{s(S_{n-1}(w)) - x}{s(S_{n-1}(w)) - y} = \frac{b + dy}{b + dx} \cdot \frac{S_{n-1}(w) - x}{S_{n-1}(w) - y} = \cdots = \left( \frac{b + dy}{b + dx} \right)^n \cdot \frac{w - x}{w - y}, \quad n \in \mathbb{N}.
\]

If \( s \) is parabolic, then by (1.5)

\[
\frac{1}{S_n(w) - y} = \frac{1}{s(S_{n-1}(w)) - y} = \frac{2d}{b + c} + \frac{1}{S_{n-1}(w) - y} = \cdots = n \cdot \frac{2d}{b + c} + \frac{1}{w - y}, \quad n \in \mathbb{N}.
\]

From this we easily get \( S_n(w) \) in closed form.

Let \( \{s_n(w)\}_{n=1}^\infty \) be a sequence of linear fractional transformations and consider the composition

\[
S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w), \quad n \in \mathbb{N}.
\]

If we now try to use (1.4)–(1.5) as in Example 1.1, the expressions get complicated, even for quite small \( n \in \mathbb{N} \), except in special cases such as (1.7).

In this paper we introduce a generalization of (1.4)–(1.5) which is adapted to this more general situation. It is based on a new concept, tail sequences, introduced in §2. These tail sequences generalize the fixed points in a certain sense. As an example we show in §3, that these formulas apply to continued fractions.

2. **Tail sequences.** Given the sequence

\[
s_n(w) = \frac{a_n + c_n w}{b_n + d_n w}, \quad \Delta_n = a_n d_n - c_n b_n \neq 0, \quad \text{for } n = 1, 2, 3, \ldots.
\]

Let us introduce the following concept:

**Definition 2.1.** \( \{u_n\}_{n=0}^\infty, u_n \in \hat{C} \), is called a tail sequence for \( \{s_n(w)\}_{n=1}^\infty \), if

\[
u_{n-1} = s_n(u_n) \quad \text{for } n = 1, 2, 3, \ldots.
\]

This concept has already been introduced for another purpose, for the special case where \( \{s_n(w)\} \) is a continued fraction generating sequence, i.e., \( c_n = 0 \) and \( d_n = 1 \) for all \( n \) [6]. We refer to §3 for information explaining the name “tail sequences”.
The following properties of tail sequences are proved by straightforward computation (we omit the proofs here):

**Proposition 2.2.** Let \( \{u_n\}_{n=0}^{\infty} \) and \( \{v_n\}_{n=0}^{\infty} \) be two tail sequences for the sequence \( \{s_n(w)\}_{n=1}^{\infty} \) of linear fractional transformations. Then

A. \[ u_n \neq v_n \text{ for an } n \in \mathbb{N} \cup \{0\} \Rightarrow u_n \neq v_n \text{ for all } n \in \mathbb{N} \cup \{0\} . \]

B. \[ \frac{1}{s_n(w) - u_{n-1}} = \frac{d_n (b_n + d_n u_n)}{-\Delta_n} + \frac{(b_n + d_n u_n)^2}{-\Delta_n} \cdot \frac{1}{w - u_n} , \quad w \in \mathcal{C} , \]
\[ \text{if } u_{n-1} \neq \infty \text{ and } u_n \neq \infty . \]

C. \[ \frac{s_n(w) - u_{n-1}}{s_n(w) - v_{n-1}} = \frac{b_n + d_n v_n}{b_n + d_n u_n} \cdot \frac{w - u_n}{w - v_n} , \quad w \in \mathcal{C} , \]
\[ \text{if } u_n, u_{n-1}, v_n, v_{n-1} \neq \infty \text{ and } u_0 \neq v_0 . \]

By repeated use of parts B and C we get the following representation of \( S_n(w) \) (\( S_n \) defined by (1.10)).

**Theorem 2.3.** Let \( \{u_n\} \) and \( \{v_n\} \) be two distinct tail sequences for the sequence \( \{s_n(w)\}_{n=1}^{\infty} \) of linear fractional transformations. Further, let

\[ S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w) \text{ for all } n \in \mathbb{N} . \]

If \( u_n \neq \infty \) for \( n = 0,1,\ldots,N \), then

\[ \frac{1}{S_n(w) - u_0} = \sum_{j=1}^{N} \frac{b_j + d_j u_j}{-\Delta_j} \prod_{k=1}^{j-1} \frac{(b_k + d_k u_k)^2}{-\Delta_k} + \frac{1}{w - u_N} \prod_{k=1}^{N} \frac{(b_k + d_k u_k)^2}{-\Delta_k} \quad \text{for all } w \in \mathcal{C} . \]

If, in addition, \( v_n \neq \infty \) for \( n = 0,1,\ldots,N \), then

\[ \frac{S_n(w) - u_0}{S_n(w) - v_0} = \frac{w - u_N}{w - v_N} \prod_{j=1}^{N} \frac{b_j + d_j u_j}{b_j + d_j v_j} \text{ for all } w \in \mathcal{C} . \]

Again the proofs are straightforward and omitted here.

From (2.3) or (2.4) we can find a closed expression for \( S_n(w) \). On the other hand, to obtain some advantage of these formulas, we need to know at least one tail sequence, \( \{u_n\} \), \( u_n \neq \infty \) for \( \{s_n(w)\} \). We shall see some examples.

**Example 2.4.** Let \( s_n = s \) for all \( n \in \mathbb{N} \), where \( s \) is given by (1.1) with \( d \neq 0 \). If \( x \) is a fixed point of \( s(w) \), then \( \{u_n\} \) with \( u_n = x \) for all \( n \) is a tail sequence for \( \{s_n(w)\} \). (All tail sequences \( \{v_n\} \) for \( \{s_n(w)\} \), where \( v_0 \) is not a fixed point of \( s \), are nonconstant.) If \( s \) has two distinct fixed points, \( x \) and \( y \) (finite since \( d \neq 0 \)), then the choice \( u_n = x \) and \( v_n = y \) in (2.4) gives (1.8). If \( s \) has only one fixed point \( y \) (s
is parabolic), then the choice \( u_n = y \) in (2.3) reduces to (1.9), since then
\[
(b - c)^2 + 4ad = 0 \quad \text{and} \quad y = (c - b)/2d.
\]
This explains why Proposition 2.2C and B represent a generalization of (1.4)–(1.5).

**Example 2.5.** Let \( s_{2n-1} = s_1 \) and \( s_{2n} = s_2 \) for all \( n \in \mathbb{N} \). Further, let \( x_{2n} = x_0 \) and \( y_{2n} = y_0 \) be two fixed points of \( S_2 = s_1 \circ s_2 \), where \( x_0 \neq y_0 \) if \( S_2 \) is nonparabolic. Then \( x_{2n+1} = x_1 = s_2(x_0) \) and \( y_{2n+1} = y_1 = s_2(y_0) \) are fixed points of the linear fractional transformation \( (s_2 \circ s_1) \), and \( \{x_n\} \) and \( \{y_n\} \) are tail sequences for \( \{s_n(w)\} \).

This procedure of finding periodic tail sequences can easily be extended to the case where \( \{s_n(w)\} \) is periodic with period \( k \in \mathbb{N} \). On the other hand we do not gain much compared to (1.8) or (1.9), since we then have
\[
S_{k_\infty + p}(w) = S_k \circ S_k \circ \cdots \circ S_k \circ S_p(w) \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad 0 < p < k.
\]

**Example 2.6.** Let \( \{u_n\}_{n=0}^\infty \), \( u_n \in \mathbb{C} \), be given. Then there exist infinitely many sequences \( \{s_n(w)\}_{n=0}^\infty \) of linear fractional transformations such that \( \{u_n\} \) is a tail sequence for \( \{s_n(w)\} \). If, for instance, \( u_n \neq 0 \) for all \( n \), then \( \{u_n\} \) is a tail sequence for \( \{2u_n u_{n-1}/(u_n + w)\} \) and for \( \{u_n(1 + u_n)/(1 + w)\} \) etc.

In some situations it suffices to find the approximate location of \( S_n(D) \) for some set \( D \subset \mathbb{C} \). This can for instance be the case if we want to prove convergence results for \( \{s_n(w)\}_{n=1}^\infty \) and estimate the truncation error for a given \( n \).

**Example 2.7.** Let
\[
(2.5) \quad s_n(w) = \frac{a_n}{1 + w} = \frac{a + \varepsilon_n}{1 + w} \quad \text{for all} \quad n \in \mathbb{N},
\]
where \( a, \varepsilon_n \in \mathbb{C} \) are chosen such that \( \Delta_n = a_n \neq 0 \) and \( s(w) = (a + cw)/(b + dw) = a/(1 + w) \) is a hyperbolic or loxodromic linear fractional transformation (i.e., \( a \neq 0, a \notin (-\infty, -1/4] \)). Let \( x \) be the attractive fixed point of \( s \), and \( y \) the repulsive one. Then \( x, y \neq \infty \) and by (1.4) \( |b + dx| > |b + dy| = |c - dx| \), i.e., here \( |1 + x| > |x| \). Let \( D = |b + dx| - |c - dx| \), i.e., here \( D = |1 + x| - |x| \), and assume that
\[
(2.6) \quad |\varepsilon_n| \leq \frac{(D^2 - \mu^2)}{4} \quad \text{for all} \quad n, 0 < \mu \leq D.
\]

Then it is possible to prove by straightforward computation that
\[
(2.7) \quad s_n(V) \subset V \quad \text{and} \quad S_n(V) \to \{u\} \subset V \quad \text{for} \quad V = \left\{ z \in \mathbb{C}; |z - x| \leq \frac{D - \mu}{2} \right\},
\]
for a \( u \in V \), and
\[
(2.8) \quad s_n^{-1}(W) \subset W \quad \text{for} \quad W = \left\{ z \in \mathbb{C}; |z - y| \leq \frac{D - \mu}{2} \right\}.
\]
(See [1].) Let \( \{u_n\} \) be the tail sequence for \( \{s_n(w)\} \) with \( u_0 = u \). Then \( u_n \in V \) for all \( n \), because the point
\[
\{u_n\} = \lim_{m \to \infty} s_{n+1} s_{n+2} \cdots s_{n+m}(V) \subset V \quad \text{for all} \quad n.
\]
Moreover, let \( \{ v_n \} \) be a tail sequence for \( \{ s_n(w) \} \) with \( v_0 \in W \). Then \( v_n \in W \) for all \( n \), since \( v_n = s_n^{-1}(v_{n-1}) \). From (2.4) we then get

\[
\frac{S_n(w) - u_0}{S_n(w) - v_0} = K_n \frac{w - u_n}{w - v_n} \quad \text{for all } w \in \mathcal{C},
\]

where

\[
|K_n| = \prod_{j=1}^{n} \left| \frac{1 + v_j}{1 + u_j} \right| \leq \prod_{j=1}^{n} \frac{|1 + y| + (D - \mu)/2}{|1 + x| - (D - \mu)/2}.
\]

Since \( u_n, v_n, u_0 \neq \infty \) and \( K_n \to 0 \), we can also solve (2.9) to get

\[
S_n(w) = \frac{w(u_0 - K_nv_0) - (u_0v_n - K_nv_0u_n)}{w(1 - K_n) - (v_n - K_nu_n)} - \frac{u_0(w - v_n)}{w - v_n}.
\]

This means in particular that \( S_n(w) \to u_0 \) for all \( w \in C \setminus W \) (which is not a new result; see the next section).

By the same method as introduced in Example 2.7, we also get estimates for \( K_n \) in (2.9) in more general cases. For instance, if

\[
s_n(w) = \frac{a + \epsilon_n + (c + \lambda_n)w}{b + dw}, \quad d > 0, \Delta_n \neq 0,
\]

where \( s \) is a hyperbolic or loxodromic linear fractional transformation, then (2.7) and (2.8) are valid for

\[V = \left\{ z \in C; |z - x| \leq \frac{D - \mu}{2d} \right\}, \quad W = \left\{ z \in C; |z - y| \leq \frac{D - \mu}{4d} \right\},\]

if

\[|\lambda_n| + 2d \frac{|\epsilon_n + x\lambda_n|}{D - \mu} \leq \frac{D - \mu}{2}, \quad |\lambda_n| \leq \frac{D - \mu}{4},\]

and

\[|\epsilon_n + y\lambda_n| \leq \frac{D^2 - \mu^2}{8d} \quad \text{for all } n.\]

This gives

\[
|K_n| = \left| \prod_{j=1}^{n} \left| \frac{b + d\lambda_j}{b + d\lambda_j} \right| \right| \leq \left| \frac{|b + dy| + (D - \mu)/4}{|b + dx| - (D - \mu)/2} \right|^{n},
\]

\[
\leq \left( \frac{|b + dx| + |c - dx| - \mu}{|b + dx| + |c - dx| + \mu} \right)^{n} \quad \text{for all } n.
\]

For \( s_n(w) = (a + \epsilon_n)/(b + \delta_n + w) \), see [2]. For even more general examples where \( V \) and \( W \) vary with \( n \), see [2 and 3].
3. Application to continued fractions. A continued fraction

\[ K_{\frac{a_n}{b_n}} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}} \quad a_n \neq 0, \]

\(a_n, b_n \in \mathbb{C}\), is an infinite process, where a sequence of approximants,

\[ f_n = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \quad \text{for} \ n = 1, 2, 3, \ldots, \]

is generated from the ordered pair \(\left(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}\right)\). We say that \(K(a_n/b_n)\) converges and has the value \(f\) if \(\lim f_n = f\) exists in \(\mathbb{C}\). Defining the linear fractional transformations

\[ s_n(w) = \frac{a_n}{b_n + w}, \quad (\Delta_n = a_n \neq 0) \quad \text{for} \ n = 1, 2, 3, \ldots, \]

we see that \(f_n = S_n(0)\), where \(S_n\) is given by (1.10).

If \(K(a_n/b_n)\) converges, then so do also all its tails

\[ K_{\frac{a_{m+n}}{b_{m+n}}} = \frac{a_{m+1}}{b_{m+1} + \frac{a_{m+2}}{b_{m+2} + \cdots}} \quad \text{for} \ m = 0, 1, 2, \ldots. \]

Let \(f^{(m)}\) denote the value of the \(m\)th tail (3.4). Then, clearly, \(f^{(n-1)} = a_n/(b_n + f^{(n)})\), that is, \(\{f^{(n)}\}\) is a tail sequence for \(\{s_n(w)\}\) or for \(K(a_n/b_n)\). This is what inspired Waadeland when choosing a name for sequences connected with \(K(a_n/b_n)\) satisfying (2.2) [6].

It is easy to prove that when \(s_n\) has the form (3.3), then \(S_n\) has the form

\[ S_n(w) = \frac{A_n + A_{n-1}w}{B_n + B_{n-1}w} \quad \text{for} \ n = 1, 2, 3, \ldots, \]

where \(\{A_n\}\) and \(\{B_n\}\) satisfy the recursion

\[ Y_n = b_nY_{n-1} + a_nY_{n-2} \quad \text{for} \ n = 1, 2, 3, \ldots, \]

with initial values \(A_{-1} = B_0 = 1\) and \(A_0 = B_{-1} = 0\). This means that the tail sequence \(\{-h_n\}\) of \(K(a_n/b_n)\) with \(-h_0 = \infty\), can be written

\[ -h_n = S_n^{-1}(\infty) = -\frac{B_n}{B_{n-1}} = -b_n - \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_2}{b_1}, \quad n \in \mathbb{N}. \]

This tail sequence is of particular importance in generating convergence results for continued fractions. From Theorem 2.3 we get the following corollary:

**Corollary 3.1.** Let \(\{u_n\}\) and \(\{v_n\}\) be two distinct tail sequences of the continued fraction \(K(a_n/b_n)\). If \(u_n, v_n \neq \infty\) for \(n = 0, 1, \ldots, N\), then

\[ S_N(0) = u_0 \left(1 - \prod_{j=0}^{N} \frac{u_j}{v_j}\right)^{-1} = u_0 \left(1 - \prod_{j=0}^{N} v_j / u_j\right)^{-1} = u_0 \frac{\prod_{j=0}^{N} v_j - \prod_{j=1}^{N} u_j}{\prod_{j=0}^{N} v_j - \prod_{j=0}^{N} u_j} \]

\[ = u_0 - u_0 \left(\sum_{m=1}^{N} \prod_{j=1}^{m-1} \left(-\frac{b_j + u_j}{u_j}\right)\right)^{-1}. \]
and

\[ h_N = -v_N \frac{1 - \prod_{j=0}^{N-1} u_j / v_j}{1 - \prod_{j=0}^{N-1} u_j / v_j} = - \frac{\prod_{j=0}^{N-1} v_j - \prod_{j=0}^{N-1} u_j}{\prod_{j=0}^{N-1} v_j - \prod_{j=0}^{N-1} u_j} = -v_N - v_N \left( \sum_{m=1}^{N} \prod_{j=m}^{N} \left( -\frac{v_j}{b_j + v_j} \right) \right)^{-1}. \]

**Proof.** $S_N(w)$ and $S_N^{-1}(w)$ are found by use of Theorem 2.3. We then use that $a_n = u_{n-1}(b_n + u_n) = v_{n-1}(b_n + v_n)$ for $n = 1, 2, \ldots, N$. □

**Remark.** If $u_n \neq \infty$, then $u_{n+1} \neq 0$ (if $n \geq 1$) and $u_{n+1} \neq -b_{n+1}$. 

**Example 2.7 (continued).** Let $s_n$ be defined as earlier. (That is, by (2.5) and such that the conditions mentioned hold.) Then $\{S_n(0)\}_{n=1}^\infty$ are the approximants of the continued fraction $K(a_n/1)$. If (2.6) holds, then $\{u_n\}$ and $\{v_n\}$ as defined earlier, are two distinct tail sequences for $\{s_n(w)\}$, that is for $K(a_n/1)$. Since $u_n \in V$ and $v_n \in W$ for all $n$, it follows that

\[ \frac{u_j}{v_j} \leq \frac{|x| + D - \mu}{2} = \frac{1 + x}{1 + x} + |x| + \mu = r < 1. \]

This means that $\prod_{j=1}^{n} (u_j/v_j)$ diverges to 0, and thereby that $\{S_n(0)\}$ converges to $u_0$ by Corollary 3.1. Hence we have proved a useful criterion for convergence of continued fractions. (This criterion is not a new result. It was proved in a different form and by a different method by Perron [4, Satz 2.40]. Later, Scott and Wall proved the parabola theorem [5] which extends this result considerably.)

This method can be used to develop new convergence criteria also. For instance, if

\[ s_n(w) = \frac{a_n + \varepsilon_n}{1 + w} \quad \text{for } n = 1, 2, 3, \ldots, \]

where $\{a_n\}$ is periodic with period length $k \in \mathbb{N}$, and the linear fractional transformation

\[ S(w) = \frac{a_1}{1 + \frac{a_2}{1 + \cdots + \frac{a_k}{1 + w}}} \]

is hyperbolic or loxodromic, the argument can be copied to prove that if $|\varepsilon_n| \leq$ some upper bound, then $K((a_n + \varepsilon_n)/1)$ converges [1]. More general results are shown in [3].

Recently, modified approximants, $S_n(w_n)$, where the $n$th tail of $K(a_n/b_n)$ is replaced by a modifying factor $w_n \in \hat{C}$, have partly replaced the ordinary approximants $f_n = S_n(0)$ in special cases. Furthermore, tail sequences $\{S_n^{-1}(u_0)\}$, where $u_0$ is not necessarily $\infty$ or the value of the continued fraction (if it converges), are found to be of interest. Clearly, Theorem 2.3 can be used for these approximants and tail sequences as well.
REFERENCES


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