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COMPOSITION OF LINEAR FRACTIONAL TRANSFORMATIONS IN TERMS OF TAIL SEQUENCES

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ABSTRACT. We consider sequences $\{s_n\}$ of linear fractional transformations. Connected to such a sequence is another sequence $\{S_n\}$ of linear fractional transformations given by

$$S_n = s_1 \circ s_2 \circ \cdots \circ s_n, \qquad n = 1, 2, 3, \ldots.$$

We introduce a new way of representing s_n (in terms of so-called tail sequences). This representation is established to give nice expressions for S_n. It can be seen as a generalization of the canonical form for s_n, which gives nice expressions for

$$T_k = \underbrace{s_n \circ s_n \circ \cdots \circ s_n}_{(k \text{ terms})}.$$

1. Introduction. A linear fractional transformation

$$s(w) = \frac{a + cw}{b + dw}, \qquad a, b, c, d \in \mathbf{C},\ \Delta = ad - bc \neq 0, \tag{1.1}$$

is a meromorphic function which maps the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ bijectively onto itself. Hence, we adopt the convention that

$$a + cw = a \quad \text{if } c = 0 \text{ and } w = \infty, \tag{1.2}$$

and

$$b + dw = b \quad \text{if } d = 0 \text{ and } w = \infty. \tag{1.3}$$

(It should be noted that the notation (1.1) used in this paper differs from the standard notation $(aw + b)/(cw + d)$.) If $s(w) \not\equiv w$, then s has two fixed points, x and y, which coincide iff s is parabolic. It is well known that then

$$\frac{s(w) - x}{s(w) - y} = \frac{b + dy}{b + dx} \cdot \frac{w - x}{w - y} \quad \text{if } x \neq y, \tag{1.4}$$

and

$$\frac{1}{s(w) - y} = \frac{2d}{b + c} + \frac{1}{w - y} \quad \text{if } x = y. \tag{1.5}$$

(We define

$$w - y = 0 \quad \text{if } w = y = \infty, \qquad w - x = 0 \quad \text{if } w = x = \infty.) \tag{1.6}$$

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Clearly, x and y are both finite iff $d \neq 0$. Hence, if $d \neq 0$, then (1.4)–(1.5) is an alternative way to write (1.1). The following example illustrates one of the advantages obtained by this way of representing s.

EXAMPLE 1.1. Given the linear fractional transformation (1.1) with $d \neq 0$ and fixed points x and y ($s(w) \not\equiv w$ since $d \neq 0$), the composition of n such transformations,

$$S_n(w) = s \circ s \circ \cdots \circ s(w), \qquad n \in \mathbf{N}, \tag{1.7}$$

is also a linear fractional transformation. If s is nonparabolic, then $x \neq y$, and by (1.4)

$$\begin{aligned} \frac{S_n(w) - x}{S_n(w) - y} &= \frac{s(S_{n-1}(w)) - x}{s(S_{n-1}(w)) - y} = \frac{b + dy}{b + dx} \cdot \frac{S_{n-1}(w) - x}{S_{n-1}(w) - y} \\ &= \cdots = \left(\frac{b + dy}{b + dx}\right)^n \cdot \frac{w - x}{w - y}, \quad n \in \mathbf{N}. \end{aligned} \tag{1.8}$$

If s is parabolic, then by (1.5)

$$\begin{aligned} \frac{1}{S_n(w) - y} &= \frac{1}{s(S_{n-1}(w)) - y} = \frac{2d}{b + c} + \frac{1}{S_{n-1}(w) - y} \\ &= \cdots = n \cdot \frac{2d}{b + c} + \frac{1}{w - y}, \qquad n \in \mathbf{N}. \end{aligned} \tag{1.9}$$

From this we easily get $S_n(w)$ in closed form.

Let $\{s_n(w)\}_{n=1}^{\infty}$ be a sequence of linear fractional transformations and consider the composition

$$S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w), \qquad n \in \mathbf{N}. \tag{1.10}$$

If we now try to use (1.4)–(1.5) as in Example 1.1, the expressions get complicated, even for quite small $n \in \mathbf{N}$, except in special cases such as (1.7).

In this paper we introduce a generalization of (1.4)–(1.5) which is adapted to this more general situation. It is based on a new concept, tail sequences, introduced in §2. These tail sequences generalize the fixed points in a certain sense. As an example we show in §3, that these formulas apply to continued fractions.

2. Tail sequences. Given the sequence

$$s_n(w) = \frac{a_n + c_n w}{b_n + d_n w}, \qquad \Delta_n = a_n d_n - c_n b_n \neq 0, \quad \text{for } n = 1, 2, 3, \ldots. \tag{2.1}$$

Let us introduce the following concept:

DEFINITION 2.1. $\{u_n\}_{n=0}^{\infty}$, $u_n \in \hat{\mathbf{C}}$, is called a tail sequence for $\{s_n(w)\}_{n=1}^{\infty}$, if

$$u_{n-1} = s_n(u_n) \quad \text{for } n = 1, 2, 3, \ldots. \tag{2.2}$$

This concept has already been introduced for another purpose, for the special case where $\{s_n(w)\}$ is a continued fraction generating sequence, i.e., $c_n = 0$ and $d_n = 1$ for all n [**6**]. We refer to §3 for information explaining the *name* "tail sequences".

The following properties of tail sequences are proved by straightforward computation (we omit the proofs here):

PROPOSITION 2.2. *Let* $\{u_n\}_{n=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$ *be two tail sequences for the sequence* $\{s_n(w)\}_{n=1}^{\infty}$ *of linear fractional transformations. Then*

A.

$$u_n \neq v_n \text{ for an } n \in \mathbf{N} \cup \{0\} \Rightarrow u_n \neq v_n \text{ for all } n \in \mathbf{N} \cup \{0\}.$$

B.

$$\frac{1}{s_n(w) - u_{n-1}} = \frac{d_n(b_n + d_n u_n)}{-\Delta_n} + \frac{(b_n + d_n u_n)^2}{-\Delta_n} \cdot \frac{1}{w - u_n}, \qquad w \in \hat{\mathbf{C}},$$

if $u_{n-1} \neq \infty$ *and* $u_n \neq \infty$.

C.

$$\frac{s_n(w) - u_{n-1}}{s_n(w) - v_{n-1}} = \frac{b_n + d_n v_n}{b_n + d_n u_n} \cdot \frac{w - u_n}{w - v_n}, \qquad w \in \hat{\mathbf{C}},$$

if $u_n, u_{n-1}, v_n, v_{n-1} \neq \infty$ *and* $u_0 \neq v_0$.

By repeated use of parts B and C we get the following representation of S_n (S_n defined by (1.10)).

THEOREM 2.3. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be two distinct tail sequences for the sequence* $\{s_n(w)\}_{n=1}^{\infty}$ *of linear fractional transformations. Further, let*

$$S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w) \quad \text{for all } n \in \mathbf{N}.$$

If $u_n \neq \infty$ *for* $n = 0, 1, \ldots, N$, *then*

$$\frac{1}{S_N(w) - u_0} = \sum_{j=1}^{N} d_j \frac{b_j + d_j u_j}{-\Delta_j} \prod_{k=1}^{j-1} \frac{(b_k + d_k u_k)^2}{-\Delta_k} + \frac{1}{w - u_N} \prod_{k=1}^{N} \frac{(b_k + d_k u_k)^2}{-\Delta_k} \tag{2.3}$$

for all $w \in \hat{\mathbf{C}}$. *If, in addition,* $v_n \neq \infty$ *for* $n = 0, 1, \ldots, N$, *then*

$$\frac{S_N(w) - u_0}{S_N(w) - v_0} = \frac{w - u_N}{w - v_N} \prod_{j=1}^{N} \frac{b_j + d_j v_j}{b_j + d_j u_j} \quad \text{for all } w \in \hat{\mathbf{C}}. \tag{2.4}$$

Again the proofs are straightforward and omitted here.

From (2.3) or (2.4) we can find a closed expression for $S_n(w)$. On the other hand, to obtain some advantage of these formulas, we need to know at least one tail sequence, $\{u_n\}$, $u_n \neq \infty$ for $\{s_n(w)\}$. We shall see some examples.

EXAMPLE 2.4. Let $s_n = s$ for all $n \in \mathbf{N}$, where s is given by (1.1) with $d \neq 0$. If x is a fixed point of $s(w)$, then $\{u_n\}$ with $u_n = x$ for all n is a tail sequence for $\{s_n(w)\}$. (All tail sequences $\{v_n\}$ for $\{s_n(w)\}$, where v_0 is *not* a fixed point of s, are nonconstant.) If s has two distinct fixed points, x and y (finite since $d \neq 0$), then the choice $u_n = x$ and $v_n = y$ in (2.4) gives (1.8). If s has only one fixed point y (s

is parabolic), then the choice $u_n = y$ in (2.3) reduces to (1.9), since then

$$(b-c)^2 + 4ad = 0 \quad \text{and} \quad y = (c-b)/2d.$$

This explains why Proposition 2.2C and B represent a generalization of (1.4)–(1.5).

EXAMPLE 2.5. Let $s_{2n-1} = s_1$ and $s_{2n} = s_2$ for all $n \in \mathbf{N}$. Further, let $x_{2n} = x_0$ and $y_{2n} = y_0$ be two fixed points of $S_2 = s_1 \circ s_2$, where $x_0 \neq y_0$ if S_2 is nonparabolic. Then $x_{2n+1} = x_1 = s_2(x_0)$ and $y_{2n+1} = y_1 = s_2(y_0)$ are fixed points of the linear fractional transformation $(s_2 \circ s_1)$, and $\{x_n\}$ and $\{y_n\}$ are tail sequences for $\{s_n(w)\}$.

This procedure of finding periodic tail sequences can easily be extended to the case where $\{s_n(w)\}$ is periodic with period $k \in \mathbf{N}$. On the other hand we do not gain much compared to (1.8) or (1.9), since we then have

$$S_{kn+p}(w) = S_k \circ S_k \circ \cdots \circ S_k \circ S_p(w) \quad \text{for } n \in \mathbf{N} \text{ and } 0 \leqslant p \leqslant k.$$

EXAMPLE 2.6. Let $\{u_n\}_{n=0}^{\infty}$, $u_n \in \mathbf{C}$, be given. Then there exist infinitely many sequences $\{s_n(w)\}_{n+1}^{\infty}$ of linear fractional transformations such that $\{u_n\}$ is a tail sequence for $\{s_n(w)\}$. If, for instance, $u_n \neq 0$ for all n, then $\{u_n\}$ is a tail sequence for $\{2u_n u_{n-1}/(u_n + w)\}$ and for $\{u_{n-1}(1+u_n)/(1+w)\}$ etc.

In some situations it suffices to find the approximate location of $S_n(D)$ for some set $D \subseteq \mathbf{C}$. This can for instance be the case if we want to prove convergence results for $\{S_n(w)\}_{n=1}^{\infty}$ and estimate the truncation error for a given n.

EXAMPLE 2.7. Let

$$s_n(w) = \frac{a_n}{1+w} = \frac{a+\varepsilon_n}{1+w} \quad \text{for all } n \in \mathbf{N}, \tag{2.5}$$

where $a, \varepsilon_n \in \mathbf{C}$ are chosen such that $\Delta_n = a_n \neq 0$ and $s(w) = (a+cw)/(b+dw) = a/(1+w)$ is a hyperbolic or loxodromic linear fractional transformation (i.e., $a \neq 0$, $a \notin (-\infty, -1/4]$). Let x be the attractive fixed point of s, and y the repulsive one. Then $x, y \neq \infty$ and by (1.4) $|b+dx| > |b+dy| = |c-dx|$, i.e., here $|1+x| > |x|$. Let $D = |b+dx| - |c-dx|$, i.e., here $D = |1+x| - |x|$, and assume that

$$|\varepsilon_n| \leqslant (D^2 - \mu^2)/4 \quad \text{for all } n, 0 < \mu \leqslant D. \tag{2.6}$$

Then it is possible to prove by straightforward computation that

(2.7)

$$s_n(V) \subseteq V \quad \text{and} \quad S_n(V) \to \{u\} \subset V \quad \text{for } V = \left\{ z \in \mathbf{C};\ |z-x| \leqslant \frac{D-\mu}{2} \right\},$$

for a $u \in V$, and

$$s_n^{-1}(W) \subseteq W \quad \text{for } W = \left\{ z \in \mathbf{C};\ |z-y| \leqslant \frac{D-\mu}{2} \right\}. \tag{2.8}$$

(See [**1**].) Let $\{u_n\}$ be the tail sequence for $\{s_n(w)\}$ with $u_0 = u$. Then $u_n \in V$ for all n, because the point

$$\{u_n\} = \lim_{m\to\infty} s_{n+1}s_{n+2} \cdots s_{n+m}(V) \subseteq V \quad \text{for all } n.$$

Moreover, let $\{v_n\}$ be a tail sequence for $\{s_n(w)\}$ with $v_0 \in W$. Then $v_n \in W$ for all n, since $v_n = s_n^{-1}(v_{n-1})$. From (2.4) we then get

$$\frac{S_n(w) - u_0}{S_n(w) - v_0} = K_n \frac{w - u_n}{w - v_n} \quad \text{for all } w \in \hat{\mathbf{C}}, \tag{2.9}$$

where

$$|K_n| = \prod_{j=1}^{n} \left| \frac{1 + v_j}{1 + u_j} \right| \leqslant \prod_{j=1}^{n} \frac{|1 + y| + (D - \mu)/2}{|1 + x| - (D - \mu)/2}$$
$$= \left(\frac{|1 + x| + |x| - \mu}{|1 + x| + |x| + \mu} \right)^n.$$

Since u_n, v_n, u_0, $v_0 \neq \infty$ and $K_n \to 0$, we can also solve (2.9) to get

$$S_n(w) = \frac{w(u_0 - K_n v_0) - (u_0 v_n - K_n v_0 u_n)}{w(1 - K_n) - (v_n - K_n u_n)} \sim \frac{u_0(w - v_n)}{w - v_n}.$$

This means in particular that $S_n(w) \to u_0$ for all $w \in \mathbf{C} \setminus W$ (which is not a new result; see the next section).

By the same method as introduced in Example 2.7, we also get estimates for K_n in (2.9) in more general cases. For instance, if

$$s_n(w) = \frac{a + \varepsilon_n + (c + \lambda_n)w}{b + dw}, \qquad d > 0, \ \Delta_n \neq 0,$$

where s is a hyperbolic or loxodromic linear fractional transformation, then (2.7) and (2.8) are valid for

$$V = \left\{ z \in \mathbf{C}; |z - x| \leqslant \frac{D - \mu}{2d} \right\}, \qquad W = \left\{ z \in \mathbf{C}; |z - y| \leqslant \frac{D - \mu}{4d} \right\},$$

if

$$|\lambda_n| + 2d \frac{|\varepsilon_n + x\lambda_n|}{D - \mu} \leqslant \frac{D - \mu}{2}, \qquad |\lambda_n| \leqslant \frac{D - \mu}{4},$$

and

$$|\varepsilon_n + y\lambda_n| \leqslant \frac{D^2 - \mu^2}{8d} \quad \text{for all } n.$$

This gives

$$|K_n| = \prod_{j=1}^{n} \left| \frac{b + dv_j}{b + du_j} \right| \leqslant \left(\frac{|b + dy| + (D - \mu)/4}{|b + dx| - (D - \mu)/2} \right)^n$$
$$\leqslant \left(\frac{|b + dx| + |c - dx| - \mu}{|b + dx| + |c - dx| + \mu} \right)^n \quad \text{for all } n.$$

For $s_n(w) = (a + \varepsilon_n)/(b + \delta_n + w)$, see **[2]**. For even more general examples where V and W vary with n, see **[2 and 3]**.

3. Application to continued fractions. A continued fraction

$$(3.1)\qquad K\frac{a_n}{b_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}, \qquad a_n \neq 0,$$

$a_n, b_n \in \mathbf{C}$, is an infinite process, where a sequence of approximants,

$$(3.2)\qquad f_n = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \quad \text{for } n = 1,2,3,\ldots,$$

is generated from the ordered pair $(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty})$. We say that $K(a_n/b_n)$ converges and has the value f if $\lim f_n = f$ exists in $\hat{\mathbf{C}}$. Defining the linear fractional transformations

$$(3.3)\qquad s_n(w) = \frac{a_n}{b_n + w}, \qquad (\Delta_n = a_n \neq 0) \quad \text{for } n = 1,2,3,\ldots,$$

we see that $f_n = S_n(0)$, where S_n is given by (1.10).

If $K(a_n/b_n)$ converges, then so do also all its tails

$$(3.4)\qquad \underset{n=1}{\overset{\infty}{K}} \frac{a_{m+n}}{b_{m+n}} = \frac{a_{m+1}}{b_{m+1}} + \frac{a_{m+2}}{b_{m+2}} + \cdots \quad \text{for } m = 0,1,2,\ldots.$$

Let $f^{(m)}$ denote the value of the mth tail (3.4). Then, clearly, $f^{(n-1)} = a_n/(b_n + f^{(n)})$, that is, $\{f^{(n)}\}$ is a tail sequence for $\{s_n(w)\}$ or for $K(a_n/b_n)$. This is what inspired Waadeland when choosing a name for sequences connected with $K(a_n/b_n)$ satisfying (2.2) **[6]**.

It is easy to prove that when s_n has the form (3.3), then S_n has the form

$$S_n(w) = \frac{A_n + A_{n-1}w}{B_n + B_{n-1}w} \quad \text{for } n = 1,2,3,\ldots,$$

where $\{A_n\}$ and $\{B_n\}$ satisfy the recursion

$$Y_n = b_n Y_{n-1} + a_n Y_{n-2} \quad \text{for } n = 1,2,3,\ldots,$$

with initial values $A_{-1} = B_0 = 1$ and $A_0 = B_{-1} = 0$. This means that the tail sequence $\{-h_n\}$ of $K(a_n/b_n)$ with $-h_0 = \infty$, can be written

$$-h_n = S_n^{-1}(\infty) = -\frac{B_n}{B_{n-1}} = -b_n - \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_2}{b_1}, \qquad n \in \mathbf{N}.$$

This tail sequence is of particular importance in generating convergence results for continued fractions. From Theorem 2.3 we get the following corollary:

COROLLARY 3.1. *Let $\{u_n\}$ and $\{v_n\}$ be two distinct tail sequences of the continued fraction $K(a_n/b_n)$. If $u_n, v_n \neq \infty$ for $n = 0,1,\ldots,N$, then*

$$S_N(0) = u_0 \frac{1 - \prod_{j=1}^{N} u_j/v_j}{1 - \prod_{j=0}^{N} u_j/v_j} = u_0 v_0 \frac{\prod_{j=1}^{N} v_j - \prod_{j=1}^{N} u_j}{\prod_{j=0}^{N} v_j - \prod_{j=0}^{N} u_j}$$

$$= u_0 - u_0 \left(\sum_{m=1}^{N+1} \prod_{j=1}^{m-1} \left(-\frac{b_j + u_j}{u_j} \right) \right)^{-1}$$

and

$$h_N = -v_N \frac{1 - \prod_{j=0}^{N} u_j/v_j}{1 - \prod_{j=0}^{N-1} u_j/v_j} = -\frac{\prod_{j=0}^{N} v_j - \prod_{j=0}^{N} u_j}{\prod_{j=0}^{N-1} v_j - \prod_{j=0}^{N-1} u_j}$$

$$= -v_N - v_N \left(\sum_{m=1}^{N} \prod_{j=m}^{N} \left(-\frac{v_j}{b_j + v_j} \right) \right)^{-1}.$$

PROOF. $S_N(w)$ and $S_N^{-1}(w)$ are found by use of Theorem 2.3. We then use that $a_n = u_{n-1}(b_n + u_n) = v_{n-1}(b_n + v_n)$ for $n = 1, 2, \ldots, N$. □

REMARK. If $u_n \neq \infty$, then $u_{n-1} \neq 0$ (if $n \geqslant 1$) and $u_{n+1} \neq -b_{n+1}$.

EXAMPLE 2.7 (CONTINUED). Let s_n be defined as earlier. (That is, by (2.5) and such that the conditions mentioned hold.) Then $\{S_n(0)\}_{n=1}^{\infty}$ are the approximants of the continued fraction $K(a_n/1)$. If (2.6) holds, then $\{u_n\}$ and $\{v_n\}$ as defined earlier, are two distinct tail sequences for $\{s_n(w)\}$, that is for $K(a_n/1)$. Since $u_n \in V$ and $v_n \in W$ for all n, it follows that

$$\left| \frac{u_j}{v_j} \right| \leqslant \frac{|x| + \dfrac{D - \mu}{2}}{|y| - \dfrac{D - \mu}{2}} = \frac{|1 + x| + |x| - \mu}{|1 + x| + |x| + \mu} = r < 1.$$

This means that $\prod_{j=1}^{\infty}(u_j/v_j)$ diverges to 0, and thereby that $\{S_N(0)\}$ converges to u_0 by Corollary 3.1. Hence we have proved a useful criterion for convergence of continued fractions. (This criterion is not a new result. It was proved in a different form and by a different method by Perron [**4**, Satz 2.40]. Later, Scott and Wall proved the parabola theorem [**5**] which extends this result considerably.)

This method can be used to develop new convergence criteria also. For instance, if

$$s_n(w) = \frac{a_n + \varepsilon_n}{1 + w} \quad \text{for } n = 1, 2, 3, \ldots,$$

where $\{a_n\}$ is periodic with period length $k \in \mathbf{N}$, and the linear fractional transformation

$$S(w) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_k}{1 + w}$$

is hyperbolic or loxodromic, the argument can be copied to prove that if $|\varepsilon_n| \leqslant$ some upper bound, then $K((a_n + \varepsilon_n)/1)$ converges [**1**]. More general results are shown in [**3**].

Recently, modified approximants, $S_n(w_n)$, where the nth tail of $K(a_n/b_n)$ is replaced by a modifying factor $w_n \in \hat{\mathbf{C}}$, have partly replaced the ordinary approximants $f_n = S_n(0)$ in special cases. Furthermore, tail sequences $\{S_n^{-1}(u_0)\}$, where u_0 is not necessarily ∞ or the value of the continued fraction (if it converges), are found to be of interest. Clearly, Theorem 2.3 can be used for these approximants and tail sequences as well.

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