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AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH A PRESCRIBED ZERO

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ABSTRACT. In this paper a new elementary proof for solving one extremal problem of real polynomials with a given real zero is given.

Let $b \geqslant 0$ and let R_n^b be a set of all polynomials $P(Z) = a_0 + \cdots + a_n Z^n$, $Z = e^{iq}$, where $a_0, \ldots, a_n$ are real coefficients, that satisfy $P(b) = 0$. For a given polynomial $P(Z) \in R_n^b$ let us introduce

$$\|P\|_c^2 = \frac{1}{2\pi}\int_0^{2\pi} |P(Z)|^2\, dq, \qquad \|P\|_L^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P(Z)}{Z-b}\right|^2 dq.$$

In [1] (see also [2]) the following extremal problem is solved:

$$\min \frac{\|P\|_c^2}{\|P\|_L^2} = 1 + b^2 - 2b\cos\frac{\pi}{n+1}, \qquad P \in R_n^b. \tag{1}$$

In this paper, using the original procedure, besides the problem (1), we solved the complementary problem:

$$\max \frac{\|P\|_c^2}{\|P\|_L^2} = 1 + b^2 + 2b\cos\frac{\pi}{n+1}, \qquad P \in R_n^b. \tag{2}$$

PROPOSITION. *For each polynomial $P(Z)$ from the set R_n^b,* (1) *and* (2) *hold. The required minimum* (*maximum*) *in* (1) ((2)) *is achieved for*

$$P(Z) = (Z-b)\sum_{k=1}^{n} C_k Z^{k-1} \sin\frac{k\pi}{n+1}, \qquad Z = e^{iq},$$

where $C_k = C$ ($C_k = (-1)^{k-1}C$), $k = 1, \ldots, n$, $C = \text{const} \neq 0$.

PROOF. Since $P(Z)$ belongs to the set R_n^b we can write it in the form

$$P(Z) = (Z-b)\left(x_1 + x_2 Z + \cdots + x_n Z^{n-1}\right), \qquad Z = e^{iq},$$

where $x_1, \ldots, x_n$ are real numbers. Now we have

$$\|P\|_L^2 = \sum_{k=1}^{n} x_k^2 \tag{3}$$

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and

$$\|P\|_c^2 = (1+b^2)\sum_{k=1}^{n} x_k^2 - 2b\sum_{k=1}^{n-1} x_k x_{k+1}. \tag{4}$$

In [3], as a special case of a general inequality, the following double inequality is proved:

$$-\cos\frac{\pi}{n+1}\sum_{k=1}^{n} x_k^2 \leqslant \sum_{k=1}^{n-1} x_k x_{k+1} \leqslant \cos\frac{\pi}{n+1}\sum_{k=1}^{n} x_k^2. \tag{5}$$

Equality holds in the left-hand inequality if and only if

$$x_k = C\sin\frac{k\pi}{n+1}, \qquad k = 1,\dots,n,$$

where $C = \text{const} \neq 0$, and in the right-hand inequality if and only if

$$x_k = (-1)^{k-1} C\sin\frac{k\pi}{n+1}, \qquad k = 1,\dots,n,$$

where $C = \text{const} \neq 0$. (On the inequality (5) see also [4–7].) From (3), (4) and (5) we obtain

$$\begin{aligned}\|P\|_c^2 &= (1+b^2)\|P\|_L^2 - 2b\sum_{k=1}^{n-1} x_k x_{k+1} \\ &\leqslant (1+b^2)\|P\|_L^2 + 2b\cos\frac{\pi}{n+1}\|P\|_L^2,\end{aligned}$$

where from (2) follows. In the same way from (3), (4) and (5) we obtain the solution of the problem (1). From equality in (5) we obtain polynomials for which the extremum in (1) and (2) is obtained.

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