PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 1, May 1986

POLE- AND ZERO-FREE REGIONS FOR ANALYTIC CONTINUED FRACTIONS

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ABSTRACT. By using well-known methods of analytic continued fraction theory, various types of zero-free regions are obtained for sequences of polynomials having complex coefficients and being defined by three-term recurrence relations. These results are related to recent investigations by P. Henrici, E. B. Saff and R. S. Varga. As an application, zero-free sectors and stripes in $\mathbf{C}$ are obtained for the Bessel function J_ν, where ν is complex. Analogous results are obtained for the Lommel polynomials associated with J_ν.

1. Introduction. Consider the continued fraction

$$(1)\qquad w_n = \frac{A_n}{B_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}, \qquad n \in \mathbf{N},$$

where $a_n, b_n \in \mathbf{C}$ and $a_n \neq 0$ for $n \in \mathbf{N}$. In the applications below, a_n and b_n are special analytic functions of z. Then (see [2, p. 20])

$$(2)\qquad \begin{aligned} B_n &= b_n B_{n-1} + a_n B_{n-2}, \quad B_0 = 1, \quad B_{-1} = 0, \qquad n \in \mathbf{N}, \\ A_n &= b_n A_{n-1} + a_n A_{n-2}, \quad A_0 = 0, \quad A_{-1} = 1, \qquad n \in \mathbf{N}, \end{aligned}$$

hold.

Because $A_n B_{n-1} - B_n A_{n-1} = (-1)^{n-1} a_1 a_2 \cdots a_n$, $n \in \mathbf{N}$, A_n and B_n do not vanish simultaneously, and, hence, $w_n = \infty$ holds iff $B_n = 0$. If a sequence B_n, $n \in \mathbf{N}$, is given which satisfies (2), then, after choosing $a_1 \in \mathbf{C}$, $a_1 \neq 0$, arbitrarily, B_n can be considered as the nth denominator of a sequence of continued fractions $w_n = A_n/B_n$ of type (1).

We want to find conditions on a_n, b_n which imply $w_n \neq \infty$ and, hence, $B_n \neq 0$ for $n \in \mathbf{N}$. Since, for $n \geqslant 2$, $w_n = a_1/(b_1 + w_{n-1}^*)$ holds, where

$$w_{n-1}^* = \frac{A_{n-1}^*}{B_{n-1}^*} = \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n},$$

$A_n = 0$ holds iff $B_{n-1}^* = 0$ holds.

Hence, results concerning the nonvanishing of B_n, $n \in \mathbf{N}$ also yield results concerning the nonvanishing of A_n, $n \in \mathbf{N}$. In [4] the special case $a_n = -\alpha_n z$, $b_n = \beta_n + z$, $\alpha_n > 0$, $\beta_n > 0$, $n \in \mathbf{N}$, is considered, and zero-free parabolic regions are obtained for the polynomials $B_n(z)$. In [1] also zero-free parabolic regions are obtained for interpolation polynomials $B_n(z)$, where $a_n = -\alpha_n z_n$, $b_n = \beta_n + z_{n+1}$,

Received by the editors May 27, 1982 and, in revised form, March 6, 1985.
1980 *Mathematics Subject Classification*. Primary 30C15; Secondary 30B70, 33A40.

$\alpha_n > 0$, $\beta_n > 0$, and $z_n = z - X_{n-1}$. Here X_n, $n \in \mathbf{N}$, are the real interpolation points. The results of [**1** and **4**] are generalized in [**3**] to sequences B_n, $n \in \mathbf{N}$, with complex α_n and β_n.

We first derive general conditions on a_n, b_n which imply $B_n \neq 0$ for $n \in \mathbf{N}$ (Lemma 1). If a_n and b_n are special polynomials in z of degree $\leqslant 2$, then Lemma 1 yields parabolic regions, sectors and stripes in $\mathbf{C}$ as zero-free regions for $B_n(z)$. In particular, zero-free regions of these types are obtained for the Bessel function J_ν and the corresponding Lommel polynomials, where ν is complex.

2. Main results. As in [**1** and **3**] the principal idea is to write (1) in the form $w_N = s_1 \circ \cdots \circ s_N(0)$, $N \in \mathbf{N}$, where $s_n(u) := a_n/(b_n + u)$, $u \in \mathbf{C}$, $n \in \mathbf{N}$, and to determine a sequence of closed half-planes $H_n \subset \overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ with $0 \in H_n$, $n \in \mathbf{N}$, such that, for $n \in \mathbf{N}$, $D_n := s_n(H_n)$ is a finite closed disk satisfying $D_n \subset H_{n-1}$ for $n \geqslant 2$. Then $w_N \in D_1$ and, hence, $w_N \neq \infty$ holds for $N \in \mathbf{N}$.

For $n \in \mathbf{N}$ define $H_n := \{z \in \mathbf{C}\colon \operatorname{Re} e^{i\varphi_n}(z + d_n) \geqslant 0\} \cup \{\infty\}$, where $\varphi_n \in \mathbf{R}$ and $d_n \in \mathbf{C}$. Then $0 \in H_n$ if $\operatorname{Re} e^{i\varphi_n} d_n \geqslant 0$, and $D_n = s_n(H_n)$ is a finite disk iff $-b_n \notin H_n$; i.e., $\operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0$.

Denote by b_n^* the point symmetric to $-b_n$ with respect to ∂H_n; that is, $b_n^* = -d_n + e^{-2i\varphi_n}(\bar{b}_n - \bar{d}_n)$. Then because $s_n(-b_n) = \infty$, the center ζ_n of D_n is given by

$$\zeta_n = s_n(b_n^*) = \frac{a_n}{b_n + b_n^*} = \frac{a_n e^{i\varphi_n}}{2 \operatorname{Re} e^{i\varphi_n}(b_n - d_n)}.$$

Since $s_n(\infty) = 0 \in \partial D_n$, the radius r_n of D_n is given by

$$r_n = |\zeta_n| = |a_n|/2 \operatorname{Re} e^{i\varphi_n}(b_n - d_n).$$

Now $D_n \subset H_{n-1}$ holds iff $\operatorname{Re} e^{i\varphi_{n-1}}(\zeta_n + d_{n-1}) - r_n \geqslant 0$ holds ($n \geqslant 2$). This is equivalent to

$$|a_n| \leqslant \operatorname{Re} e^{i(\varphi_{n-1}+\varphi_n)} a_n + 2\left(\operatorname{Re} e^{i\varphi_{n-1}} d_{n-1}\right)\left(\operatorname{Re} e^{i\varphi_n}(b_n - d_n)\right), \qquad n \geqslant 2.$$

We have thus proved (see also [**2**, Chapter 4])

LEMMA 1. *Assume that a_n and b_n in* (1) *or* (2) *satisfy* (*for suitable $\varphi_n \in \mathbf{R}$, $d_n \in \mathbf{C}$*)

$$\operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0, \quad \operatorname{Re} e^{i\varphi_n} d_n \geqslant 0, \qquad n \in \mathbf{N}, \tag{3}$$

and

$$|a_n| \leqslant \operatorname{Re} e^{i(\varphi_{n-1}+\varphi_n)} a_n + 2\left(\operatorname{Re} e^{i\varphi_{n-1}} d_{n-1}\right)\left(\operatorname{Re} e^{i\varphi_n}(b_n - d_n)\right), \qquad n \geqslant 2. \tag{4}$$

Then $B_n \neq 0$ for $n \in \mathbf{N}$. If (3) *holds for $1 \leqslant n \leqslant N$ and* (4) *holds for $2 \leqslant n \leqslant N$, then $B_n \neq 0$ for $1 \leqslant n \leqslant N$.*

REMARK. In (3) the condition $\operatorname{Re} e^{i\varphi_1}(b_1 - d_1) > 0$ can be replaced by the weaker condition $-b_1 \notin D_2$—i.e.

$$\left|2b_1 \operatorname{Re} e^{i\varphi_2}(b_2 - d_2) + a_2 e^{i\varphi_2}\right| > |a_2|.$$

We now consider special cases of (1) and Lemma 1.

THEOREM 1. *Let*

$$w_n(z) = \frac{A_n(z)}{B_n(z)} = \frac{1}{1} + \frac{c_1c_2z}{1} + \frac{c_2c_3z}{1} + \cdots + \frac{c_{n-1}c_nz}{1},$$

$$B_n(z) = B_{n-1}(z) + c_{n-1}c_nzB_{n-2}(z), \quad n \geqslant 1, \qquad B_0 = 1, \quad B_{-1} = 0,$$

where $c_n \in \mathbf{C}$, $c_n = |c_n|e^{i\psi_n} \neq 0$, $\alpha \leqslant \psi_n < \beta$, $\beta - \alpha < \pi$, $n \in \mathbf{N}$.

(a) *Then* $B_n(z) \neq 0$ *for* $n \geqslant 1$ *and all* $z = |z|e^{i\vartheta}$ *satisfying* $-\pi - 2\alpha < \vartheta \leqslant \pi - 2\beta$.

(b) *If* $I := \{\varphi \in \mathbf{R}\colon \beta - \pi/2 \leqslant \varphi < \alpha + \pi/2\}$,

$$r(\varphi) := \inf_{n \geqslant 2} \operatorname{Re}(e^{i\varphi}/c_{n-1})\operatorname{Re}(e^{i\varphi}/c_n) \quad \textit{for } \varphi \in I,$$

and if, for $r \geqslant 0$, $P(r,\varphi) := \{z \in \mathbf{C}\colon |z| \leqslant \operatorname{Re} e^{2i\varphi}z + r/2\}$, *then* $B_n(z) \neq 0$ *for* $n \geqslant 1$ *and all* $z \in \bigcup_{\varphi \in I} P(r(\varphi), \varphi)$.

PROOF. Put $a_1 = 1$, $a_n = c_{n-1}c_nz$, $n \geqslant 2$, $b_n = 1$, $n \geqslant 1$, in (1) and apply Lemma 1. (a) Choose $d_n = 0$ and $e^{i\varphi_n} = e^{-i\vartheta/2}|c_n|/c_n$, $n \geqslant 1$. Then (4) is trivially satisfied, and (3) reduces to $\operatorname{Re} e^{i\vartheta/2}c_n > 0$, $n \geqslant 1$, which is satisfied if $-\pi/2 < \psi_n + \vartheta/2 < \pi/2$, $n \geqslant 1$, or $-\pi - 2\alpha < \vartheta \leqslant \pi - 2\beta$.

(b) Choose $d_n = 1/2$ and $e^{i\varphi_n} = e^{i\varphi}|c_n|/c_n$, $n \geqslant 1$. Then (3) and (4) reduce to $\operatorname{Re} e^{i\varphi}/c_n > 0$ and $|z| \leqslant \operatorname{Re} e^{2i\varphi}z + \frac{1}{2}\operatorname{Re}(e^{i\varphi}/c_{n-1})\operatorname{Re}(e^{i\varphi}/c_n)$. $\operatorname{Re} e^{i\varphi}/c_n > 0$ is satisfied if $-\pi/2 < \varphi - \psi_n < \pi/2$, $n \geqslant 1$, or $\beta - \pi/2 \leqslant \varphi < \alpha + \pi/2$.

THEOREM 2. *Let*

$$w_n(z) = \frac{A_n(z)}{B_n(z)} = \frac{1}{1 + e_1z} + \frac{c_1c_2z^2}{1 + e_2z} + \frac{c_2c_3z^2}{1 + e_3z} + \cdots + \frac{c_{n-1}c_nz^2}{1 + e_nz},$$

$$B_n(z) = (1 + e_nz)B_{n-1}(z) + c_{n-1}c_nz^2B_{n-2}(z), \quad n \geqslant 1, \qquad B_0 = 1, \quad B_{-1} = 0,$$

where e_n, $c_n \in \mathbf{C}$ *and* $c_n = |c_n|e^{i\psi_n} \neq 0$, $\alpha \leqslant \psi_n < \beta$, $\beta - \alpha < \pi$ *for* $n \in \mathbf{N}$.

(a) *If* $\operatorname{Re}(e_n/c_n) \geqslant 0$ $(\leqslant 0)$, $n \geqslant 1$, *then* $B_n(z) \neq 0$ *for* $n \geqslant 1$ *and all* $z = |z|e^{i\vartheta}$ *satisfying* $-\alpha - \pi/2 < \vartheta \leqslant -\beta + \pi/2$ $(-\alpha + \pi/2 < \vartheta \leqslant -\beta + 3\pi/2)$.

(b) *If* $e_n = 0$, $n \geqslant 1$, *and* I, $r(\varphi)$ *are the same as in Theorem* 1, *and if, for* $r \geqslant 0$, $S(r,\varphi) := \{z \in \mathbf{C}\colon |\operatorname{Im} e^{i\varphi}z| \leqslant \sqrt{r}/2\}$, *then* $B_n(z) \neq 0$ *for* $n \geqslant 1$ *and all* $z \in \bigcup_{\varphi \in I} S(r(\varphi), \varphi)$.

PROOF. Put $a_1 = 1$, $a_n = c_{n-1}c_nz^2$, $n \geqslant 2$, $b_n = 1 + e_nz$, $n \geqslant 1$, in (1) and apply Lemma 1. (a) Choose $d_n = e_nz$ and $e^{i\varphi_n} = e^{-i\vartheta}|c_n|/c_n$, $n \geqslant 1$. Then (4) is trivially satisfied, and (3) reduces to $\operatorname{Re}(e_n/c_n) \geqslant 0$ and $\operatorname{Re} zc_n > 0$, $n \geqslant 1$, which is satisfied if $-\pi/2 < \psi_n + \vartheta < \pi/2$, $n \geqslant 1$, or $-\alpha - \pi/2 < \vartheta \leqslant -\beta + \pi/2$. Similarly, $e^{i\varphi_n} = -e^{-i\vartheta}|c_n|/c_n$ yields the conditions $\operatorname{Re}(e_n/c_n) \leqslant 0$ and $\operatorname{Re}(-zc_n) > 0$ or $-\pi/2 < -\pi + \psi_n + \vartheta < \pi/2$, $n \geqslant 1$, which is satisfied if $-\alpha + \pi/2 < \vartheta \leqslant -\beta + 3\pi/2$.

(b) Put $d_n = 1/2$ and $e^{i\varphi_n} = e^{i\varphi}|c_n|/c_n$. Then (4) and (3) reduce to

$$|z^2| \leqslant \operatorname{Re}(e^{i\varphi}z)^2 + \operatorname{Re}(e^{i\varphi}/c_{n-1})\operatorname{Re}(e^{i\varphi}/c_n)/2$$

or $2|\operatorname{Im} e^{i\varphi}z| \leqslant (\operatorname{Re}(e^{i\varphi}/c_{n-1})\operatorname{Re}(e^{i\varphi}/c_n))^{1/2}$, $n \geqslant 2$, and $\operatorname{Re} e^{i\varphi}/c_n > 0$, $n \geqslant 1$, which is satisfied if $-\pi/2 < \varphi - \psi_n < \pi/2$, $n \geqslant 1$ or $\beta - \pi/2 \leqslant \varphi < \alpha + \pi/2$.

THEOREM 3. *Let*

$$w_n(z) = \frac{A_n(z)}{B_n(z)} = \frac{1}{g_1 z + h_1} + \frac{1}{g_2 z + h_2} + \cdots + \frac{1}{g_n z + h_n},$$

$$B_n(z) = (g_n z + h_n) B_{n-1}(z) + B_{n-2}(z), \quad n \geqslant 1, \qquad B_0 = 1, \quad B_{-1} = 0,$$

where g_n, $h_n \in \mathbf{C}$, $g_n = |g_n| e^{i\psi_n} \neq 0$, $\alpha \leqslant \psi_n < \beta$, $\beta - \alpha < \pi$ *for* $n \in \mathbf{N}$.

(a) *If* $\operatorname{Re} h_n \geqslant 0$ $(\leqslant 0)$, $n \geqslant 1$, *then* $B_n(z) \neq 0$ *for* $n \geqslant 1$ *and all* $z = |z| e^{i\vartheta} \neq 0$ *satisfying* $-\alpha - \pi/2 < \vartheta \leqslant -\beta + \pi/2$ $(-\alpha + \pi/2 < \vartheta \leqslant -\beta + 3\pi/2)$.

(b) *If* $h_n = 0$, $n \geqslant 1$,

$$J := \{\varphi \in \mathbf{R}: -\alpha - \pi/2 < \varphi \leqslant -\beta + \pi/2\},$$

$$s(\varphi) := \inf_{n \geqslant 2} \operatorname{Re}(e^{i\varphi} g_{n-1}) \operatorname{Re}(e^{i\varphi} g_n)$$

for $\varphi \in J$, *and if*

$$D(0, \varphi) := \{z \in \mathbf{C}: \operatorname{Im} e^{-i\varphi} z = 0\}$$

and

$$D(s, \varphi) := \{z \in \mathbf{C}: |z - ie^{i\varphi}/\sqrt{s}| \geqslant 1/\sqrt{s}\} \cap \{z \in \mathbf{C}: |z + ie^{i\varphi}/\sqrt{s}| \geqslant 1/\sqrt{s}\}$$

for $s > 0$, *then* $B_n(z) \neq 0$ *for* $n \geqslant 1$ *and all* $z \in \bigcup_{\varphi \in J} D(s(\varphi), \varphi) \setminus \{0\}$.

PROOF. Put $a_n = 1$, $b_n = g_n z + h_n$, $n \geqslant 1$, in (1) and apply Lemma 1.

(a) Choose $d_n = 0$, $n \geqslant 1$. Then (3) and (4) reduce to $\operatorname{Re} e^{i\varphi_n}(g_n z + h_n) > 0$, $n \geqslant 1$, and $1 \leqslant \cos(\varphi_{n-1} + \varphi_n)$, $n \geqslant 2$. This holds if $\varphi_n = 0$, $\operatorname{Re} h_n \geqslant 0$ and $\operatorname{Re} z g_n > 0$ or $-\pi/2 < \vartheta + \psi_n < \pi/2$, $n \geqslant 1$, which is satisfied if $-\alpha - \pi/2 < \vartheta \leqslant -\beta + \pi/2$. Similarly, $\varphi_n = \pi$, $n \geqslant 1$, yields the conditions $\operatorname{Re} h_n \leqslant 0$ and $\operatorname{Re}(-z g_n) > 0$ or $-\pi/2 < -\pi + \vartheta + \psi_n < \pi/2$, $n \geqslant 1$, which is satisfied if $-\alpha + \pi/2 < \vartheta \leqslant -\beta + 3\pi/2$.

(b) Put $d_n = z g_n / 2$ and $e^{i\varphi_n} = e^{i\varphi} |z| / z$, $n \geqslant 1$. Then (4) and (3) reduce to

$$1 \leqslant \operatorname{Re}(\bar{z} e^{i\varphi}/|z|)^2 + |z|^2 \operatorname{Re}(e^{i\varphi} g_{n-1}) \operatorname{Re}(e^{i\varphi} g_n)/2, \qquad n \geqslant 2,$$

and

$$\operatorname{Re}(e^{i\varphi} g_n) > 0 \quad \text{or} \quad -\pi/2 < \varphi + \psi_n < \pi/2, \qquad n \geqslant 1,$$

which is satisfied if $-\alpha - \pi/2 < \varphi \leqslant -\beta + \pi/2$. Finally, one verifies that for $s \geqslant 0$,

$$\{z \in \mathbf{C}: |z|^2 \leqslant \operatorname{Re}(z e^{-i\varphi})^2 + |z|^4 s/2\} = D(s, \varphi).$$

REMARK. If the assumptions in Theorems 1–3 hold for $1 \leqslant n \leqslant N$ only, then the conclusions are valid for $1 \leqslant n \leqslant N$, provided $r(\varphi)$, $s(\varphi)$ are replaced by

$$r_N(\varphi) := \min_{2 \leqslant n \leqslant N} \operatorname{Re}\left(\frac{e^{i\varphi}}{c_{n-1}}\right) \operatorname{Re}\left(\frac{e^{i\varphi}}{c_n}\right) \quad \text{for } \varphi \in I,$$

and

$$s_N(\varphi) := \min_{2 \leqslant n \leqslant N} \operatorname{Re}(e^{i\varphi} g_{n-1}) \operatorname{Re}(e^{i\varphi} g_n) \quad \text{for } \varphi \in J.$$

3. Application to Bessel functions and Lommel polynomials. Let $J_\nu(z) = (z/2)^\nu \phi_\nu(z)$ be the νth Bessel function, where

$$\phi_\nu(z) := \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)}, \qquad \nu, z \in \mathbf{C}, \nu \neq -1, -2, \ldots.$$

Then (see also [**5**, Chapter III])

$$\phi_\nu'(z) = -(z/2)\phi_{\nu+1}(z), \tag{5}$$

$$\phi_\nu(z) = (\nu+1)\phi_{\nu+1}(z) - (z/2)^2\phi_{\nu+2}(z), \tag{6}$$

$$z\phi_\nu''(z) + (2\nu+1)\phi_\nu'(z) + z\phi_\nu(z) = 0. \tag{7}$$

Here (7) implies that all ϕ_ν have simple zeros. Then (5) shows that ϕ_ν, $\phi_{\nu+1}$ have no common zeros. Using [**2**, Theorem 5.16], we obtain from (6) that

$$\frac{\phi_{\nu+1}(z)}{\phi_\nu(z)} = \frac{1}{\nu+1} - \frac{(z/2)^2}{\nu+2} - \frac{(z/2)^2}{\nu+3} - \cdots, \tag{8}$$

where the infinite continued fraction is uniformly convergent on each compact set of $\mathbf{C}$ avoiding the poles of $\phi_{\nu+1}/\phi_\nu$—i.e., the zeros of ϕ_ν. After applying an equivalence transformation to (8), we obtain

$$\frac{(\nu+1)\phi_{\nu+1}(z)}{\phi_\nu(z)} = \frac{1}{1} + \frac{c_1c_2z^2}{1} + \frac{c_2c_3z^2}{1} + \cdots, \tag{9}$$

where $c_n = 1/2i(\nu+n)$, $n \geqslant 1$. Applying Theorem 2 to the sequence of approximants of (9) or, equivalently, of (8) yields

THEOREM 4. *If* $\nu = \nu_1 + i\nu_2$, $\nu_1, \nu_2 \in \mathbf{R}$, *such that* $0 \leqslant \arg(\nu+1) < \pi$, *and if*

$$B_{n,\nu}(z) = (\nu+n)B_{n-1,\nu}(z) - (z/2)^2 B_{n-2,\nu}(z), \qquad n \geqslant 1,$$
$$B_{0,\nu} = 1, \qquad B_{-1,\nu} = 0,$$

then $B_{n,\nu}(z) \neq 0$ *for* $n \geqslant 1$ *and*

(a) *all* $z = |z|e^{i\vartheta}$ *satisfying* $\arg(\nu+1) < \vartheta \leqslant \pi$ *or* $\arg(\nu+1) + \pi < \vartheta \leqslant 2\pi$,

(b) *all* $z \in \mathbf{C}$ *satisfying* $|\operatorname{Im} z| \leqslant \nu_2$, *provided* $\nu_2 > 0$.

Also,

(c) *if* $0 \leqslant \arg(\nu+1) < \pi/2$, *then, in addition to* (a) *and* (b), $B_{n,\nu}(z) \neq 0$ *for* $n \geqslant 1$ *and all* $z \in \mathbf{C}$ *satisfying* $(\operatorname{Re} z)^2 \leqslant (\nu_1+1)(\nu_1+2)$.

Because $B_{n,\bar\nu}(z) = \overline{B_{n,\nu}(\bar z)}$, *the zeros of* $B_{n,\bar\nu}$ *are the conjugates of the zeros of* $B_{n,\nu}$.

PROOF. With the notations of Theorem 2 we obtain $e_n = 0$, $n \geqslant 1$, $\alpha = -\arg(\nu+1) - \pi/2 \leqslant \psi_n < \beta = -\pi/2$ and $\beta - \alpha = \arg(\nu+1)$. Then (a) follows from Theorem 2(a). Furthermore,

$$I = \{\varphi\colon -\pi \leqslant \varphi < -\arg(\nu+1)\}$$

and

$$r(\varphi) = \inf_{n \geqslant 2} 4\operatorname{Re}(e^{i\varphi}i(\nu+n-1))\operatorname{Re}(e^{i\varphi}i(\nu+n)) \quad \text{for } \varphi \in I.$$

Then Theorem 2(b) implies $B_{n,\nu}(z) \neq 0$ for $n \geqslant 1$ and all $z \in \bigcup_{\varphi \in I} S(r(\varphi), \varphi)$. Choosing especially $\varphi = -\pi$ and $\varphi = -\pi/2$ yields (b) and (c).

COROLLARY 1. *If* $\nu = \nu_1 + i\nu_2$, $\nu_1, \nu_2 \in \mathbf{R}$, *such that* $0 \leqslant \arg(\nu + 1) < \pi$, *then* $\phi_\nu(z) \neq 0$ *for*

(a) *all* $z = |z|e^{i\vartheta}$ *satisfying* $\arg(\nu + 1) < \vartheta < \pi$ *or* $\arg(\nu + 1) + \pi < \vartheta < 2\pi$,

(b) *all* $z \in \mathbf{C}$ *satisfying* $|\operatorname{Im} z| < \nu_2$, *provided* $\nu_2 > 0$.

Also,

(c) *if* $0 \leqslant \arg(\nu + 1) < \pi/2$, *then, in addition to* (a) *and* (b), $\phi_\nu(z) \neq 0$ *for all* $z \in \mathbf{C}$ *satisfying* $(\operatorname{Re} z)^2 < (\nu_1 + 1)(\nu_1 + 2)$. (*See also* [**5**, *p.* 486] *for the special case* $\nu + 1 > 0$.)

Because $\phi_{\bar\nu}(z) = \overline{\phi_\nu(\bar z)}$, *the zeros of* $\phi_{\bar\nu}$ *are the conjugates of the zeros of* ϕ_ν.

REMARK. Because of the asymptotic formula [**5**, p. 199]

$$J_\nu(z) \sim (2/\pi z)^{1/2}\cos(z - \nu\pi/2 - \pi/4),$$

valid for $z \to \infty$, $-\pi < \arg z < \pi$, the "large" zeros of J_ν are near the lines $\operatorname{Im} z = \operatorname{Im} \nu\pi/2$ for $\operatorname{Re} z > 0$ or $\operatorname{Im} z = -\operatorname{Im} \nu\pi/2$ for $\operatorname{Re} z < 0$, provided $\operatorname{Im} \nu > 0$. See also [**5**, pp. 495–497].

Next, the Lommel polynomials $g_{m,\nu}(z)$, being related to $B_{m,\nu}(z)$, are defined as follows [**5**, p. 303]:

$$g_{m,\nu}(z) := \sum_{0 \leqslant n \leqslant m/2} (-1)^n \binom{m-n}{n} z^n \frac{\Gamma(\nu + m - n + 1)}{\Gamma(\nu + n + 1)},$$

$$\nu, z \in \mathbf{C}, \ \nu \neq -1, -2, \ldots.$$

Then

(10)

$$g_{m,\nu}(z) = (\nu + m) g_{m-1,\nu}(z) - z g_{m-2,\nu}(z), \quad m \geqslant 1, \qquad g_{0,\nu} = 1, \quad g_{-1,\nu} = 0.$$

Therefore $g_{m,\nu}$ and $g_{m-1,\nu}$ have no common zeros. Furthermore by [**5**, p. 302], $\lim_{m\to\infty} g_{m,\nu}(z^2/4)/\Gamma(\nu + m + 1) = \phi_\nu(z)$ uniformly on each compact set in $\mathbf{C}$. (10) yields

$$\frac{g_{m-1,\nu}(z)}{g_{m,\nu}(z)} = \frac{1}{\nu + m -} \frac{z}{\nu + m - 1 -} \cdots \frac{z}{-\nu + 1}$$

or, after applying an equivalence transformation,

$$(11) \qquad \frac{(\nu + m) g_{m-1,\nu}(z)}{g_{m,\nu}(z)} = \frac{1}{1 +} \frac{c_1 c_2 z}{1} + \cdots + \frac{c_{m-1} c_m z}{1},$$

where $c_n = 1/i(\nu + m + 1 - n)$, $1 \leqslant n \leqslant m$.

Applying Theorem 1 to (11) yields

THEOREM 5. *If* $\nu = \nu_1 + i\nu_2$, $\nu_1, \nu_2 \in \mathbf{R}$, *such that* $0 \leqslant \arg(\nu + 1) < \pi$, *then for each* $m \in \mathbf{N}$, $g_{m,\nu}(z) \neq 0$ *for*

(a) *all* $z = |z|e^{i\vartheta}$ *satisfying* $2\arg(\nu + 1) < \vartheta < 2\pi + 2\arg(\nu + m)$,

(b) *all* $z \in \mathbf{C}$ *satisfying* $|z| \leqslant \operatorname{Re} z + \nu_2^2/2$.

Also,

(c) *if* $0 \leqslant \arg(\nu + 1) < \pi/2$, *then, in addition to* (a), (b), $g_{m,\nu}(z) \neq 0$ *for all* $z \in \mathbf{C}$ *satisfying* $|z| \leqslant -\operatorname{Re} z + (\nu_1 + 1)(\nu_1 + 2)/2$.

Because $g_{m,\bar\nu}(z) = \overline{g_{m,\nu}(\bar z)}$, *the zeros of* $g_{m,\bar\nu}$ *are the conjugates of the zeros of* $g_{m,\nu}$.

PROOF. With the notation of Theorem 1, we obtain, for $1 \leqslant n \leqslant m$, $\alpha = -\arg(\nu + 1) - \pi/2 \leqslant \psi_n \leqslant \beta = -\arg(\nu + m) - \pi/2$. Then (a) follows from Theorem 1(a). Furthermore,

$$I = \{\varphi\colon -\arg(\nu + m) - \pi < \varphi < -\arg(\nu + 1)\}$$

and

$$r(\varphi) = \min_{2 \leqslant n \leqslant m} \operatorname{Re}(e^{i\varphi} i(\nu + n - 1)) \operatorname{Re}(e^{i\varphi} i(\nu + n)) \quad \text{for } \varphi \in I.$$

Then Theorem 1(b) implies $g_{m,\nu}(z) \neq 0$ for all $z \in \bigcup_{\varphi \in I} P(r(\varphi), \varphi)$. Choosing especially $\varphi = -\pi$ and $\varphi = -\pi/2$ yields (b) and (c).

REMARK. Using (9) and applying Theorem 2 to the infinite continued fraction (with arbitrary $c_\nu, e_\nu \in \mathbf{C}$, $c_\nu \neq 0$, $1 \leqslant \nu \leqslant n$)

$$\frac{1}{1 + e_1 z} + \frac{c_1 c_2 z^2}{1 + e_2 z} + \cdots + \frac{c_{n-1} c_n z^2}{1 + e_n z + c_n c_{n+1} z^2 (\nu + 1) \phi_{\nu+1}(z)/\phi_\nu(z)}$$
$$= \frac{(\nu + 1) c_n c_{n+1} z^2 A_{n-1}(z) \phi_{\nu+1}(z) + A_n(z) \phi_\nu(z)}{(\nu + 1) c_n c_{n+1} z^2 B_{n-1}(z) \phi_{\nu+1}(z) + B_n(z) \phi_\nu(z)},$$

one can obtain information on the location of zeros of

$$(\nu + 1) c_n c_{n+1} z^2 B_{n-1}(z) \phi_{\nu+1}(z) + B_n(z) \phi_\nu(z)$$

or, using (5), $B_n(z)\phi_\nu(z) - 2(\nu + 1) c_n c_{n+1} z B_{n-1}(z) \phi'_\nu(z)$.

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