POLE- AND ZERO-FREE REGIONS
FOR ANALYTIC CONTINUED FRACTIONS

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Abstract. By using well-known methods of analytic continued fraction theory, various types of zero-free regions are obtained for sequences of polynomials having complex coefficients and being defined by three-term recurrence relations. These results are related to recent investigations by P. Henrici, E. B. Saff and R. S. Varga. As an application, zero-free sectors and stripes in $\mathbb{C}$ are obtained for the Bessel function $J_v$, where $v$ is complex. Analogous results are obtained for the Lommel polynomials associated with $J_v$.

1. Introduction. Consider the continued fraction

$$w_n = \frac{A_n}{B_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2 + \cdots + b_n}, \quad n \in \mathbb{N},$$

where $a_n, b_n \in \mathbb{C}$ and $a_n \neq 0$ for $n \in \mathbb{N}$. In the applications below, $a_n$ and $b_n$ are special analytic functions of $z$. Then (see [2, p. 20])

$$B_n = b_n B_{n-1} + a_n B_{n-2}, \quad B_0 = 1, \quad B_{-1} = 0, \quad n \in \mathbb{N},$$

$$A_n = b_n A_{n-1} + a_n A_{n-2}, \quad A_0 = 0, \quad A_{-1} = 1, \quad n \in \mathbb{N},$$

hold.

Because $A_n B_{n-1} - B_n A_{n-1} = (-1)^{n-1} a_1 a_2 \cdots a_n$, $n \in \mathbb{N}$, $A_n$ and $B_n$ do not vanish simultaneously, and, hence, $w_n = \infty$ holds iff $B_n = 0$. If a sequence $B_n$, $n \in \mathbb{N}$, is given which satisfies (2), then, after choosing $a_1 \in \mathbb{C}$, $a_1 \neq 0$, arbitrarily, $B_n$ can be considered as the $n$th denominator of a sequence of continued fractions $w_n = A_n/B_n$ of type (1).

We want to find conditions on $a_n$, $b_n$ which imply $w_n \neq \infty$ and, hence, $B_n \neq 0$ for $n \in \mathbb{N}$. Since, for $n \geq 2$, $w_n = a_1/(b_1 + w_{n-1})$ holds, where

$$w_{n-1}^{*} = \frac{A_{n-1}^{*}}{B_{n-1}^{*}} = \frac{a_2}{b_2 + \cdots + b_n},$$

$A_n = 0$ holds iff $B_{n-1}^{*} = 0$ holds.

Hence, results concerning the nonvanishing of $B_n$, $n \in \mathbb{N}$ also yield results concerning the nonvanishing of $A_n$, $n \in \mathbb{N}$. In [4] the special case $a_n = -\alpha_n z$, $b_n = \beta_n + z$, $\alpha_n > 0$, $\beta_n > 0$, $n \in \mathbb{N}$, is considered, and zero-free parabolic regions are obtained for the polynomials $B_n(z)$. In [1] also zero-free parabolic regions are obtained for interpolation polynomials $B_n(z)$, where $a_n = -\alpha_n z_n$, $b_n = \beta_n + z_{n+1}$,
\( \alpha_n > 0, \beta_n > 0, \) and \( z_n = z - X_{n-1} \). Here \( X_n, n \in \mathbb{N} \), are the real interpolation points. The results of \([1\) and \(4]\) are generalized in \([3]\) to sequences \( B_n, n \in \mathbb{N} \), with complex \( \alpha_n \) and \( \beta_n \).

We first derive general conditions on \( a_n, b_n \) which imply \( B_n \neq 0 \) for \( n \in \mathbb{N} \) \( (\text{Lemma 1}) \). If \( a_n \) and \( b_n \) are special polynomials in \( z \) of degree \( \leq 2 \), then Lemma 1 yields parabolic regions, sectors and stripes in \( \mathbb{C} \) as zero-free regions for \( B_n(z) \). In particular, zero-free regions of these types are obtained for the Bessel function \( J_\nu \) and the corresponding Lommel polynomials, where \( \nu \) is complex.

2. Main results. As in \([1\) and \(3]\) the principal idea is to write \((1)\) in the form \( w_N = s_1 \circ \cdots \circ s_N(0), N \in \mathbb{N}, \) where \( s_n(u) := a_n/(b_n + u), u \in \mathbb{C}, n \in \mathbb{N}, \) and to determine a sequence of closed half-planes \( H_n \subset \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) with \( 0 \in H_n, n \in \mathbb{N} \), such that, for \( n \in \mathbb{N}, D_n := s_n(H_n) \) is a finite closed disk satisfying \( D_n \subseteq H_{n-1} \) for \( n \geq 2 \). Then \( w_n \in D_1 \) and, hence, \( w_N \neq \infty \) holds for \( N \in \mathbb{N} \).

For \( n \in \mathbb{N} \) define \( H_n := \{ z \in \mathbb{C}: \text{Re}\{e^{i\varphi_n}(z + d_n)\} \geq 0 \} \cup \{\infty\}, \) where \( \varphi_n \in \mathbb{R} \) and \( d_n \in \mathbb{C} \). Then \( 0 \in H_n \) if \( \text{Re}\{e^{i\varphi_n}d_n\} \geq 0 \), and \( D_n := s_n(H_n) \) is a finite disk iff \( -b_n \notin H_n \), i.e., \( \text{Re}\{e^{i\varphi_n}(b_n - d_n)\} > 0 \).

Denote by \( b_n^* \) the point symmetric to \( -b_n \) with respect to \( \partial H_n \); that is, \( b_n^* = -d_n + e^{-2i\varphi_n}(b_n - d_n). \) Then because \( s_n(-b_n) = \infty \), the center \( \xi_n \) of \( D_n \) is given by

\[
\xi_n = s_n(b_n^*) = \frac{a_n}{b_n + b_n^*} = \frac{a_n}{2 \text{Re}\{e^{i\varphi_n}(b_n - d_n)\}}.
\]

Since \( s_n(\infty) = 0 \in \partial D_n \), the radius \( r_n \) of \( D_n \) is given by

\[
r_n = |\xi_n| = |a_n|/2 \text{Re}\{e^{i\varphi_n}(b_n - d_n)\}.
\]

Now \( D_n \subset H_{n-1} \) holds iff \( \text{Re}\{e^{i\varphi_n}(\xi_n + d_{n-1})\} - r_n \geq 0 \) holds \( (n \geq 2) \). This is equivalent to

\[
|a_n| \leq \text{Re}\{e^{i(\varphi_n + \varphi_{n-1})}a_n + 2\text{Re}\{e^{i\varphi_{n-1}}d_{n-1}\}(\text{Re}\{e^{i\varphi_n}(b_n - d_n)\}), \quad n \geq 2.
\]

We have thus proved (see also \([2, \text{Chapter 4}]\))

\textbf{Lemma 1.} Assume that \( a_n \) and \( b_n \) in \((1)\) or \((2)\) satisfy \( (\text{for suitable } \varphi_n \in \mathbb{R}, 
\end{align*}
\]
\[
(3) \quad \text{Re}\{e^{i\varphi_n}(b_n - d_n)\} > 0, \quad \text{Re}\{e^{i\varphi_n}d_n\} \geq 0, \quad n \in \mathbb{N},
\]
and

\[
(4) \quad |a_n| \leq \text{Re}\{e^{i(\varphi_n + \varphi_{n-1})}a_n + 2(\text{Re}\{e^{i\varphi_{n-1}}d_{n-1}\})(\text{Re}\{e^{i\varphi_n}(b_n - d_n)\}), \quad n \geq 2.
\]

Then \( B_n \neq 0 \) for \( n \in \mathbb{N} \). If \((3)\) holds for \( 1 \leq n \leq N \) and \((4)\) holds for \( 2 \leq n \leq N \), then \( B_n \neq 0 \) for \( 1 \leq n \leq N \).

\textbf{Remark.} In \((3)\) the condition \( \text{Re}\{e^{i\varphi_1}(b_1 - d_1)\} > 0 \) can be replaced by the weaker condition \( -b_1 \notin D_2 \) — i.e.,

\[
|2b_1 \text{Re}\{e^{i\varphi_2}(b_2 - d_2) + a_2e^{i\varphi_2}\}| > |a_2|.
\]

We now consider special cases of \((1)\) and Lemma 1.
Theorem 1. Let
\[
\frac{w_n(z)}{B_n(z)} = \frac{A_n(z)}{B_n(z)} = \frac{1}{1 + \frac{c_1c_2z}{1 + \frac{c_2c_3z}{1 + \cdots + \frac{c_{n-1}c_nz}{1}}}},
\]
\[B_n(z) = B_{n-1}(z) + c_{n-1}c_nzB_{n-2}(z), \quad n \geq 1, \quad B_0 = 1, \quad B_1 = 0,
\]
where \(c_n \in \mathbb{C}, \ c_n = |c_n|e^{i\psi_n} \neq 0, \alpha \leq \psi_n < \beta, \beta - \alpha < \pi, n \in \mathbb{N}.
\]
(a) Then \(B_n(z) \neq 0\) for \(n \geq 1\) and all \(z = |z|e^{i\theta}\) satisfying \(-\pi - 2\alpha < \theta \leq \pi - 2\beta\).
(b) If \(I := \{\varphi \in \mathbb{R}: \beta - \pi/2 < \varphi < \alpha + \pi/2\},\)
\[r(\varphi) := \inf_{n \geq 2} \Re(e^{i\varphi}/c_{n-1})\Re(e^{i\varphi}/c_n) \quad \text{for } \varphi \in I,
\]
and if, for \(r \geq 0, \ P(\varphi, \varphi) := \{z \in \mathbb{C}: |z| \leq \Re e^{2i\varphi}z + r/2\}, \text{ then } B_n(z) \neq 0 \text{ for } n \geq 1 \text{ and all } z \in \bigcup_{\varphi \in I} P(\varphi, \varphi).
\]

Proof. Put \(a_1 = 1, \ a_n = c_{n-1}c_nz, \ n \geq 2, \ b_n = 1, \ n \geq 1,\) in (1) and apply Lemma 1. (a) Choose \(d_n = 0\) and \(e^{i\varphi_n} = e^{-i\theta}c_n/c_n, \ n \geq 1.\) Then (4) is trivially satisfied, and (3) reduces to \(\Re e^{i\varphi}/c_n > 0, \ n \geq 1,\) which is satisfied if \(-\pi/2 < \psi_n + \theta/2 \leq \pi/2, \ n \geq 1, \) or \(-\pi - 2\alpha < \theta \leq \pi - 2\beta\).

(b) Choose \(d_n = 1/2\) and \(e^{i\varphi_n} = e^{i\theta}|c_n|/c_n, \ n \geq 1.\) Then (3) and (4) reduce to \(\Re e^{i\varphi}/c_n > 0\) and \(|z| \leq \Re e^{2i\varphi}z + 1/2 \Re(e^{i\varphi}/c_{n-1})\Re(e^{i\varphi}/c_n). \) Re\(e^{i\varphi}/c_n > 0\) is satisfied if \(-\pi/2 < \theta - \psi_n < \pi/2, \ n \geq 1, \) or \(\beta - \pi/2 < \varphi < \alpha + \pi/2.
\]

Theorem 2. Let
\[
\frac{w_n(z)}{B_n(z)} = \frac{A_n(z)}{B_n(z)} = \frac{1}{1 + \frac{c_1c_2z^2}{1 + \frac{c_2c_3z^2}{1 + \cdots + \frac{c_{n-1}c_nz^2}{1}}}},
\]
\[B_n(z) = (1 + e_nz)B_{n-1}(z) + c_{n-1}c_nz^2B_{n-2}(z), \quad n \geq 1, \quad B_0 = 1, \quad B_1 = 0,
\]
where \(e_n, c_n \in \mathbb{C} \text{ and } c_n = |c_n|e^{i\psi_n} \neq 0, \alpha \leq \psi_n < \beta, \beta - \alpha < \pi \text{ for } n \in \mathbb{N}.
\]
(a) If \(\Re(e_n/c_n) \geq 0 \leq 0, \ n \geq 1, \) then \(B_n(z) \neq 0 \text{ for } n \geq 1 \text{ and all } z = |z|e^{i\theta} \text{ satisfying } -\pi/2 < \psi_n + \theta < \pi/2, \ n \geq 1, \) or \(-\alpha - \pi/2 < \theta < -\beta + \pi/2. \)

(b) If \(e_n = 0, \ n \geq 1, \) and \(I, r(\varphi) \) are the same as in Theorem 1, and if, for \(r \geq 0, \ S(\varphi, \varphi) := \{z \in \mathbb{C}: |\Im e^{i\varphi}z| \leq \sqrt{r/2}\}, \text{ then } B_n(z) \neq 0 \text{ for } n \geq 1 \text{ and all } z \in \bigcup_{\varphi \in I} S(\varphi, \varphi).
\]

Proof. Put \(a_1 = 1, \ a_n = c_{n-1}c_nz^2, \ n \geq 2, \ b_n = 1 + e_nz, \ n \geq 1,\) in (1) and apply Lemma 1. (a) Choose \(d_n = e_nz\) and \(e^{i\varphi_n} = e^{-i\theta}|c_n|/c_n, \ n \geq 1.\) Then (4) is trivially satisfied, and (3) reduces to \(\Re(e_n/c_n) \geq 0 \text{ and } \Re e_nz > 0, \ n \geq 1, \) which is satisfied if \(-\pi/2 < \psi_n + \theta < \pi/2, \ n \geq 1, \) or \(-\alpha - \pi/2 < \theta < -\beta + \pi/2. \)

Similarly, \(e^{i\varphi_n} = -e^{-i\theta}|c_n|/c_n\) yields the conditions \(\Re(e_n/c_n) \leq 0\) and \(\Re(e_nz) > 0\) or \(-\pi/2 < -\pi + \psi_n + \theta < \pi/2, \ n \geq 1, \) which is satisfied if \(-\alpha + \pi/2 < \theta < -\beta + 3\pi/2.
\]

(b) Put \(d_n = 1/2\) and \(e^{i\varphi_n} = e^{i\theta}|c_n|/c_n. \) Then (4) and (3) reduce to
\[|z|^2 \leq \Re(e^{i\varphi}z)^2 + \Re(e^{i\varphi}/c_{n-1})\Re(e^{i\varphi}/c_n)/2
\]
or \(2|\Im e^{i\varphi}z| \leq (\Re(e^{i\varphi}/c_{n-1})\Re(e^{i\varphi}/c_n))^{1/2}, \ n \geq 2, \) and \(\Re(e^{i\varphi}/c_n) > 0, \ n \geq 1, \) which is satisfied if \(-\pi/2 < \varphi - \psi_n < \pi/2, \ n \geq 1 \text{ or } \beta - \pi/2 < \varphi < \alpha + \pi/2.\)
Theorem 3. Let
\[ w_n(z) = \frac{A_n(z)}{B_n(z)} = \frac{1}{g_1 z + h_1} + \frac{1}{g_2 z + h_2} + \cdots + \frac{1}{g_n z + h_n}, \]
\[ B_n(z) = (g_n z + h_n) B_{n-1}(z) + B_{n-2}(z), \quad n \geq 1, \quad B_0 = 1, \quad B_{-1} = 0, \]
where \( g_n, h_n \in \mathbb{C}, \ g_n = |g_n|e^{i\psi_n} \neq 0, \ \alpha < \psi_n < \beta, \ \beta - \alpha < \pi \) for \( n \in \mathbb{N} \).

(a) If \( \text{Re} h_n \geq 0 (\leq 0), \ n \geq 1, \) then \( B_n(z) \neq 0 \) for \( n \geq 1 \) and all \( z = |z|e^{i\theta} \neq 0 \) satisfying \( -\alpha - \pi/2 < \theta < -\beta + \pi/2 \) (\( -\alpha + \pi/2 < \theta < -\beta + 3\pi/2 \)).

(b) If \( h_n = 0, \ n \geq 1, \)
\[ J := \{ \varphi \in \mathbb{R}: -\alpha - \pi/2 < \varphi < -\beta + \pi/2 \}, \]
\[ s(\varphi) := \inf_{n \geq 2} \text{Re}(e^{i\varphi g_{n-1}})\text{Re}(e^{i\varphi g_n}) \]
for \( \varphi \in J \), and if
\[ D(0, \varphi) := \{ z \in \mathbb{C}: \text{Im} e^{-i\varphi z} = 0 \} \]
and
\[ D(s, \varphi) := \{ z \in \mathbb{C}: |z - ie^\varphi/\sqrt{s}| \geq 1/\sqrt{s} \} \cap \{ z \in \mathbb{C}: |z + ie^\varphi/\sqrt{s}| \geq 1/\sqrt{s} \} \]
for \( s > 0 \), then \( B_n(z) \neq 0 \) for \( n \geq 1 \) and all \( z \in \cup_{\varphi \in J} D(s(\varphi), \varphi) \setminus \{0\} \).

Proof. Put \( a_n = 1, b_n = g_n z + h_n, n \geq 1 \), in (1) and apply Lemma 1.

(a) Choose \( d_n = 0, n \geq 1 \). Then (3) and (4) reduce to \( \text{Re} e^{i\varphi_n}(g_n z + h_n) > 0, \ n \geq 1, \) and \( 1 \leq \cos(\varphi_{n-1} + \varphi_n), \ n \geq 2 \). This holds if \( \varphi_n = 0, \ \text{Re} h_n \geq 0 \) and \( \text{Re} g_n > 0 \) or \( -\pi/2 < \theta + \psi_n < \pi/2, \ n \geq 1 \), which is satisfied if \( -\alpha - \pi/2 < \theta < -\beta + \pi/2 \). Similarly, \( \varphi_n = \pi, \ n \geq 1 \), yields the conditions \( \text{Re} h_n \leq 0 \) and \( \text{Re}(-g_n) > 0 \) or \( -\pi/2 < -\pi + \theta + \psi_n < \pi/2, \ n \geq 1 \), which is satisfied if \( -\alpha + \pi/2 < \theta < -\beta + 3\pi/2 \).

(b) Put \( d_n = z g_n/2 \) and \( e^{i\varphi_n} = e^i|z|/z, \ n \geq 1 \). Then (4) and (3) reduce to
\[ 1 \leq \text{Re}(\bar{z} e^{i\varphi} |z|^2) + |z|^2 \text{Re}(e^{i\varphi g_{n-1}})\text{Re}(e^{i\varphi g_n})/2, \quad n \geq 2, \]
and
\[ \text{Re}(e^{i\varphi g_n}) > 0 \ or \ -\pi/2 < \varphi + \psi_n < \pi/2, \quad n \geq 1, \]
which is satisfied if \( -\alpha - \pi/2 < \varphi < -\beta + \pi/2 \). Finally, one verifies that for \( s \geq 0, \)
\[ \{ z \in \mathbb{C}: |z|^2 \leq \text{Re}(ze^{-i\varphi})^2 + |z|^4 s/2 \} = D(s, \varphi). \]

Remark. If the assumptions in Theorems 1–3 hold for \( 1 \leq n \leq N \) only, then the conclusions are valid for \( 1 \leq n \leq N \), provided \( r(\varphi), s(\varphi) \) are replaced by
\[ r_N(\varphi) := \min_{2 \leq n \leq N} \text{Re}\left(\frac{e^{i\varphi}}{c_{n-1}}\right)\text{Re}\left(\frac{e^{i\varphi}}{c_n}\right) \]
for \( \varphi \in I \),
and
\[ s_N(\varphi) := \min_{2 \leq n \leq N} \text{Re}(e^{i\varphi g_{n-1}})\text{Re}(e^{i\varphi g_n}) \]
for \( \varphi \in J \).
3. Application to Bessel functions and Lommel polynomials. Let $J_{\nu}(z) = (z/2)^{\nu}\phi_{\nu}(z)$ be the $\nu$th Bessel function, where

$$\phi_{\nu}(z) := \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n!\Gamma(\nu + n + 1)}, \quad \nu, z \in \mathbb{C}, \nu \neq -1, -2, \ldots.$$ 

Then (see also [5, Chapter III])

(5) $\phi_{\nu}(z) = -(z/2)^{\nu+1}(z)$,

(6) $\phi_{\nu}(z) = (\nu + 1)\phi_{\nu+1}(z) - (z/2)^2\phi_{\nu+2}(z),$

(7) $z\phi''_{\nu}(z) + (2\nu + 1)\phi'_{\nu}(z) + z\phi_{\nu}(z) = 0.$

Here (7) implies that all $\phi_{\nu}$ have simple zeros. Then (5) shows that $\phi_{\nu}$, $\phi_{\nu+1}$ have no common zeros. Using [2, Theorem 5.16], we obtain from (6) that

$$\frac{\phi_{\nu+1}(z)}{\phi_{\nu}(z)} = \frac{1}{\nu + 1} - \frac{(z/2)^2}{\nu + 2} - \frac{(z/2)^2}{\nu + 3} - \cdots,$$

where the infinite continued fraction is uniformly convergent on each compact set of $\mathbb{C}$ avoiding the poles of $\phi_{\nu+1}/\phi_{\nu}$—i.e., the zeros of $\phi_{\nu}$. After applying an equivalence transformation to (8), we obtain

$$\frac{(\nu + 1)\phi_{\nu+1}(z)}{\phi_{\nu}(z)} = \frac{1}{1 + \frac{c_1c_2z^2}{1 + \frac{c_2c_3z^2}{1 + \cdots}},$$

where $c_n = 1/2i(\nu + n)$, $n \geq 1$. Applying Theorem 2 to the sequence of approximants of (9) or, equivalently, of (8) yields

**Theorem 4.** If $\nu = \nu_1 + iv_2$, $\nu_1, v_2 \in \mathbb{R}$, such that $0 \leq \arg(\nu + 1) < \pi$, and if

$$B_{n,\nu}(z) = (\nu + n)B_{n-1,\nu}(z) - (z/2)^2B_{n-2,\nu}(z), \quad n \geq 1,$$

$$B_{0,\nu} = 1, \quad B_{-1,\nu} = 0,$$

then $B_{n,\nu}(z) \neq 0$ for $n \geq 1$ and

(a) all $z = |z|e^{i\theta}$ satisfying $\arg(\nu + 1) < \theta \leq \pi$ or $\arg(\nu + 1) + \pi < \theta \leq 2\pi$,

(b) all $z \in \mathbb{C}$ satisfying $|\text{Im} z| \leq v_2$, provided $v_2 > 0$.

Also,

(c) if $0 \leq \arg(\nu + 1) < \pi/2$, then, in addition to (a) and (b), $B_{n,\nu}(z) \neq 0$ for $n \geq 1$ and all $z \in \mathbb{C}$ satisfying $|\text{Re} z|^2 \leq (\nu + 1)(\nu_1 + 2)$.

Because $B_{n,\nu}(z) = \overline{B_{n,\nu}(\bar{z})}$, the zeros of $B_{n,\nu}$ are the conjugates of the zeros of $B_{n,\nu}$.

**Proof.** With the notations of Theorem 2 we obtain $e_n = 0$, $n \geq 1$, $\alpha = -\arg(\nu + 1) - \pi/2 < \psi_n < \beta = -\pi/2$ and $\beta - \alpha = \arg(\nu + 1)$. Then (a) follows from Theorem 2(a). Furthermore,

$$I = \{ \varphi: -\pi < \varphi < -\arg(\nu + 1) \}$$

and

$$r(\varphi) = \inf_{n \geq 2} 4\text{Re}(e^{i\varphi}(\nu + n - 1))\text{Re}(e^{i\varphi}(\nu + n)) \quad \text{for } \varphi \in I.$$ 

Then Theorem 2(b) implies $B_{n,\nu}(z) \neq 0$ for $n \geq 1$ and all $z \in \bigcup_{\varphi \in I} S(r(\varphi), \varphi)$. Choosing especially $\varphi = -\pi$ and $\varphi = -\pi/2$ yields (b) and (c).
Corollary 1. If $\nu = \nu_1 + i\nu_2$, $\nu_1, \nu_2 \in \mathbf{R}$, such that $0 \leq \arg(\nu + 1) < \pi$, then $\phi_\nu(z) \neq 0$ for

(a) all $z = |z|e^{i\theta}$ satisfying $\arg(\nu + 1) < \theta < \pi$ or $\arg(\nu + 1) + \pi < \theta < 2\pi$,
(b) all $z \in \mathbf{C}$ satisfying $|\text{Im} z| < \nu_2$, provided $\nu_2 > 0$.

Also,

(c) if $0 \leq \arg(\nu + 1) < \pi/2$, then, in addition to (a) and (b), $\phi_\nu(z) \neq 0$ for all $z \in \mathbf{C}$ satisfying $(\text{Re} z)^2 < (\nu_1 + 1)(\nu_1 + 2)$. (See also [5, p. 486] for the special case $\nu + 1 > 0$.)

Because $\phi_\nu(z) = \phi_{\bar{\nu}}(\bar{z})$, the zeros of $\phi_\nu$ are the conjugates of the zeros of $\phi_{\bar{\nu}}$.

Remark. Because of the asymptotic formula [5, p. 199]

$$J_\nu(z) \sim (2/\pi z)^{1/2} \cos(z - \nu\pi/2 - \pi/4),$$

valid for $z \to \infty$, $-\pi < \arg z < \pi$, the "large" zeros of $J_\nu$ are near the lines $\text{Im} z = \text{Im} \nu\pi/2$ for $\text{Re} z > 0$ or $\text{Im} z = -\text{Im} \nu\pi/2$ for $\text{Re} z < 0$, provided $\text{Im} \nu > 0$. See also [5, pp. 495–497].

Next, the Lommel polynomials $g_{m,\nu}(z)$, being related to $B_{m,\nu}(z)$, are defined as follows [5, p. 303]:

$$g_{m,\nu}(z) := \sum_{0 \leq n \leq m/2} (-1)^n \binom{m}{n} z^n \frac{\Gamma(\nu + m - n + 1)}{\Gamma(\nu + n + 1)}.$$  

Therefore $g_{m,\nu}$ and $g_{m-1,\nu}$ have no common zeros. Furthermore by [5, p. 302],

$$\lim_{m \to \infty} g_{m,\nu}(z^2/4)/\Gamma(\nu + m + 1) = \phi_\nu(z)$$

uniformly on each compact set in $\mathbf{C}$. (10) yields

$$\frac{g_{m-1,\nu}(z)}{g_{m,\nu}(z)} = \frac{1}{\nu + m} \frac{z}{\nu + m - 1 - \cdots - \nu + 1}$$

or, after applying an equivalence transformation,

$$\frac{(\nu + m)g_{m-1,\nu}(z)}{g_{m,\nu}(z)} = \frac{1}{1 + \frac{c_1 c_2 z}{1} + \cdots + \frac{c_{m-1} c_m z}{1}},$$

where $c_n = 1/i(\nu + m + 1 - n), 1 \leq n \leq m$.

Applying Theorem 1 to (11) yields

Theorem 5. If $\nu = \nu_1 + i\nu_2$, $\nu_1, \nu_2 \in \mathbf{R}$, such that $0 \leq \arg(\nu + 1) < \pi$, then for each $m \in \mathbf{N}$, $g_{m,\nu}(z) \neq 0$ for

(a) all $z = |z|e^{i\theta}$ satisfying $2\arg(\nu + 1) < \theta < 2\pi + 2\arg(\nu + m)$,
(b) all $z \in \mathbf{C}$ satisfying $|z| \leq \text{Re} z + \nu_2^2/2$.

Also,

(c) if $0 \leq \arg(\nu + 1) < \pi/2$, then, in addition to (a), (b), $g_{m,\nu}(z) \neq 0$ for all $z \in \mathbf{C}$ satisfying $|z| \leq -\text{Re} z + (\nu_1 + 1)(\nu_1 + 2)/2$.

Because $g_{m,\nu}(z) = g_{m,\nu}(\bar{z})$, the zeros of $g_{m,\nu}$ are the conjugates of the zeros of $g_{m,\nu}$. 
Proof. With the notation of Theorem 1, we obtain, for $1 \leq n \leq m$, $\alpha = -\arg(\nu + 1) - \pi/2 < \psi_n \leq \beta = -\arg(\nu + m) - \pi/2$. Then (a) follows from Theorem 1(a). Furthermore,

$$I = \{ \varphi: -\arg(\nu + m) - \pi < \varphi < -\arg(\nu + 1) \}$$

and

$$r(\varphi) = \min_{2 \leq n \leq m} \Re(e^{i\varphi}(\nu + n - 1)) \Re(e^{i\varphi}(\nu + n)) \quad \text{for } \varphi \in I.$$

Then Theorem 1(b) implies $g_{m,n}(z) \neq 0$ for all $z \in \bigcup_{\varphi \in I} P(r(\varphi), \varphi)$. Choosing especially $\varphi = -\pi$ and $\varphi = -\pi/2$ yields (b) and (c).

Remark. Using (9) and applying Theorem 2 to the infinite continued fraction (with arbitrary $c_\nu, e_\nu \in \mathbb{C}, c_\nu \neq 0, 1 \leq \nu \leq n$)

$$1 + c_1 c_2 z^2 + \cdots + c_{n-1} c_n z^2$$

one can obtain information on the location of zeros of

$$(\nu + 1)c_n c_{n+1} z^2 B_{n-1}(z) \phi_{n+1}(z) + 2(\nu + 1)c_n c_{n+1} z B_{n-1}(z) \phi(n)(z),$$

or, using (5),

$$(\nu + 1)c_n c_{n+1} z^2 B_{n-1}(z) \phi_{n+1}(z) + B_n(z) \phi_n(z).$$

References


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