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COMPLETE METRICS CONFORMAL TO THE HYPERBOLIC DISC

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ABSTRACT. In this paper we study complete metrics conformal to the hyperbolic disc. We show that any smooth function K bounded between two negative constants is the curvature of such a metric. We also show that if $K \geq 0$ near the boundary, K cannot be the curvature of such a metric.

Introduction. Here we consider complete metrics which are conformally equivalent to the hyperbolic metric on the unit disc in $\mathbf{R}^2$. If we denote the hyperbolic metric by $h = dx\,dy/(1-r^2)^2$ and if g is conformally equivalent to h, then g is of the form

$$g = \frac{e^{2u}\,dx\,dy}{(1-r^2)^2}$$

for some function u. If we denote the Gauss curvature of g by K_g, then u satisfies the differential equation

$$(*) \qquad \Delta_h u = -1 - K_g e^{2u},$$

where Δ_h is the hyperbolic Laplacian

$$\Delta_h = (1-r^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

which henceforth we denote by Δ.

We pose the question: For which smooth functions K can equation $(*)$ be solved so that the resulting metric is complete? We also discuss the uniqueness of the solution for a given K.

We can summarize our results as follows:

Existence. Suppose there exist complete metrics g_1, g_2 conformal to h with $g_1 \leq g_2$ and $K_{g_1} \leq K_{g_2}$. Let $K_{g_1} \leq K \leq K_{g_2}$ be given. Then there exists a complete metric g with $g_1 \leq g \leq g_2$ and with Gauss curvature equal to K. In particular, if $c_2 \leq K \leq c_1 < 0$, then there is a complete metric uniformly and conformally equivalent to h with curvature equal to K. In this case, the metric will also be unique.

We will exhibit examples of complete metrics with curvature tending to $-\infty$ at the boundary, complete metrics with curvature tending to 0 at the boundary, and metrics which have some positive curvature.

Nonexistence. There are no complete metrics conformally equivalent to h with curvature nonnegative in a neighborhood of the boundary.

Uniqueness. If K is bounded between two negative constants, then the solution to $(*)$ is unique. However, there exist functions K which tend to a negative constant

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near the boundary and for which there are distinct uniformly equivalent conformal metrics with curvature equal to K.

Boundedness. If $K \leq c < 0$ in a neighborhood of the boundary, then the solution is bounded from above, while if $K \geq -N$ in a neighborhood of the boundary the solution is bounded below. These follow readily from the local Schwarz lemma [**2**]. We will also show that if $K \to 0$ near the boundary, there is no complete metric uniformly and conformally equivalent to h with curvature equal to K. The same result holds if $K \to -\infty$ near the boundary.

Similar results on the existence and uniqueness of solutions have been independently obtained by P. Aviles and R. McOwen [**1**]; they have also extended their results to the case $n \geq 3$.

1. Existence.

THEOREM 1. *Suppose ψ, ϕ and K are smooth functions on $B = \{x \in \mathbf{R}^2 \mid |x| < 1\}$ with the properties that*

(i) $\phi \leq \psi$ in B, and

(ii) $\Delta\phi \geq -1 - K(x)e^{2\phi}$, $\Delta\psi \leq -1 - K(x)e^{2\psi}$.

Then there exists $u \in C^\infty(B)$ with $\phi \leq u \leq \psi$ such that $\Delta u = -1 - K(x)e^{2u}$.

PROOF. Fix a smooth function $\phi \leq g \leq \psi$ on B. For $R < 1$, set $B_R = \{x \in \mathbf{R}^2 \mid |\mathbf{x}| < R\}$. On B_R the operator Δ is uniformly elliptic. Thus by standard elliptic theory (see, for example, [**3**]) we can find $u_R \in C^\infty(B_R)$ satisfying

$$\Delta u_R = -1 - K(x)e^{2u_R} \quad \text{in } B_R \text{ and } \phi \leq u_R \leq \psi,$$

$$u_R|_{\partial B_R} = g|_{\partial B_R}.$$

We show that a subsequence of the u_R converges uniformly on compact sets to a smooth solution u.

It is clear that $\phi \leq u_R \leq \psi$ on $B_{R'}$ for all $R > R'$. We now proceed to get uniform gradient estimates on $B_{R'}$ for all u_R, $R \geq R'' = R' + \frac{1}{2}(1 - R')$.

It is well known that the inequality

$$|d_{x,\partial B_{R''}} |\nabla u_R|\,|_{0,B_{R''}} \leq C(|u_R|_{0,B_{R''}} + |d^2_{x,\partial B_{R''}} \Delta_E u_R|_{0,B_{R''}})$$

holds, where C is a constant depending only on dimension, Δ_E is the Laplacian with respect to the Euclidean (flat) metric, $|\nabla u|$ is the norm with respect to the Euclidean metric and $d_{x,\partial B_R}$ denotes the distance from x to the boundary of B_R. Applying this to u_R for $x \in B_{R'}$, one obtains

$$\begin{aligned}
\tfrac{1}{2}(1 - R')|\nabla u_R|_{0,B_{R'}} &\leq |d_{x,\partial B_{R''}} |\nabla u_R|\,|_{0,B_{R''}} \\
&\leq C(|u_R|_{0,B_{R''}} + |d^2_{x,\partial B_{R''}} \Delta_E u_R|_{0,B_{R''}}) \\
&\leq C(|u_R|_{0,B_{R''}} + |d^2_{x,\partial B} \Delta_E u_R|_{0,B_{R''}}).
\end{aligned}$$

We note that $d^2_{x,\partial B} \Delta_E u_R$ is uniformly equivalent to Δu_R. Since u_R satisfies $(*)$ in B_R, the right-hand side of the inequality is bounded by

$$C(|u_R|_{0,B_{R''}} + |-1 - K(x)e^{2u_R}|_{0,B_{R''}})$$

which in turn is bounded independently of R. Thus we can let $R \to 1$ and we get a uniform C^1-bound on u_R in $u_{R'}$ for $R > R' + \frac{1}{2}(1 - R')$. Standard elliptic theory now gives us uniform estimates on the $C^{2+\alpha}$ norm of u_R independent of R.

Let R_j be an increasing sequence of numbers $0 < R_j < R_{j+1} < 1$ tending to 1. On B_{R_1} the sequence $\{u_{R_j}\}_{j=2}^{\infty}$ is uniformly bounded in $|\ |_{2+\alpha, B_{R_1}}$ and hence has a subsequence $(u_{1,j})$ which converges to a solution. On B_{R_2}, the sequence $u_{1,j}$ are uniformly bounded in the $|\ |_{2+\alpha, B_{R_2}}$ norm (for j sufficiently large). Hence, we extract a subsequence $u_{2,j}$ which converges uniformly on B_{R_2}. Continue inductively in this fashion. Then a subsequence of the diagonal sequence will converge uniformly on compact subsets to a solution. $\square$

COROLLARY. *Suppose that $c_1 \le K \le c_2 < 0$. Then there exists a smooth function u which solves $\Delta u = -1 - K(x)e^{2u}$.*

We now give examples of metrics $e^{2u}h$ which exhibit, in turn, the following properties:

(i) $K > 0$ at some points,
(ii) $K \to -\infty$ as $r \to 1$,
(iii) $K \to 0$ as $r \to 1$.

We will see in §2 that any metric which exhibits property (ii) or (iii) cannot be uniformly equivalent to h. Thus the solutions cannot be bounded.

(i), (iii). Let $\phi = (1 - r^2)^p$. Then one computes that $\Delta\phi = \phi(4p^2r^2 - 4p)$. If $g = e^{2\phi}h$, then

$$K_g = -(\Delta\phi + 1)e^{-2\phi} = -(1 + \phi(4p^2r^2 - 4p))e^{-2(1-r^2)^p}.$$

If $p = 1$, then g is a complete metric with positive curvature at 0. If $p = -1$, then $g = e^{2/1-r^2}h$ is a complete metric and $K_g = -(1 + \phi(4r^2 + 4))e^{-2/1-r^2}$ tends to 0 at $r = 1$.

(ii) Let $\phi \in C^\infty(B)$ have the property that $\phi = C \neq 0$ near $r = 0$ and $\phi = -\ln(-\ln(1 - r^2))$ near $r = 1$. Then near $r = 1$,

$$\Delta\phi = 4[r^2(\ln(1 - r^2))^{-2} + (\ln(1 - r^2))^{-1}]$$

and if $g = e^{2\phi}h$,

$$K_g = -[1 + 4\{r^2(\ln(1 - r^2))^{-2} + (\ln(1 - r^2))^{-1}\}](\ln(1 - r^2))^2$$

which tends to $-\infty$.

We see that g is complete since

$$g_{ij} = (1/u \ln u)^2 \delta_{ij}$$

for $u = (1 - r^2)$.

2. Nonexistence.

THEOREM 2. *Suppose $K \in C^\infty(B)$ is nonnegative in a neighborhood of ∂B. Then there does not exist any complete metric g, conformal to h, having curvature K.*

PROOF. Suppose $g = e^{2u}h$ is such a metric. Consider the identity mapping $(B, g) \to (B, h)$. By the local Schwarz lemma [**2**], the map is metric decreasing, up to a factor. Thus $e^{2u}/(1 - r^2)^2 \ge C/(1 - r^2)^2$ and u is bounded below.

Thus we may assume that $u \geq C$ satisfies $\Delta u = -1 - K(x)e^{2u}$. We denote by $\bar{u}(r)$ the average of u on circles of radius r. Since Δ is rotationally symmetric, $\overline{\Delta u} = \Delta\bar{u}$. As K is nonnegative near the boundary, it follows that $\Delta\bar{u} \leq -1$ near ∂B. Thus

$$\frac{(1-r^2)^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{u}}{\partial r}\right) \leq -1.$$

We may integrate this inequality explicitly to see that $\bar{u} \to -\infty$. Hence there is a sequence $p_i \to \partial B$ such that $u(p_i) \to -\infty$, contradicting the lower boundedness of u. □

PROPOSITION 3. *If $K \to 0$ or $K \to -\infty$ near ∂B, there do not exist any conformal metrics uniformly equivalent to h, with curvature K. (That is, there do not exist any bounded solutions to $(*)$.)*

PROOF. First suppose $K \to 0$ near ∂B and $g = e^{2u}h$ has curvature K. Thus $\Delta u \to -1$ near ∂B. In particular $\Delta u \leq c < 0$ near ∂B. By radial averaging as in the proof of Theorem 2, we see that u is not bounded below.

Now suppose that $K \to -\infty$ near ∂B and $g = e^{2u}h$ has curvature K. Since $K = -(\Delta u + 1)e^{-2u}$ we see that $\Delta u \to +\infty$. Using the radial averaging technique again, we see that there is a sequence p_i with $u(p_i) \to \infty$. □

REMARK. It is an immediate consequence of the local Schwarz lemma that (i) if $K \leq -\varepsilon$ near ∂B, then u is bounded above, and (ii) if $K \geq -N$ near ∂B, then u is bounded below.

3. Uniqueness.

PROPOSITION 4. *Let $c_1 \leq K \leq c_2 < 0$. Then there exists a unique complete metric g conformal to h, with curvature K.*

PROOF. Existence has already been shown. Thus let $g_1 = e^{2u}h$ and $g_2 = e^{2v}h$ both have curvature K. Then $\Delta(u-v) = -K(e^{2u} - e^{2v})$. By the remark above, u and v are both bounded. Thus $(u-v)$ is bounded. By the generalized maximum principle [**4**], there is a sequence of points p_k for which $\lim_{k\to\infty}(u-v)(p_k) = \inf_B(u-v)$ and $\lim_{k\to\infty}\Delta(u-v) \geq 0$. Thus $\lim_{k\to\infty}(-K(e^{2u}-e^{2v})) \geq 0$ implying $\lim_{k\to\infty}(u-v) \geq 0$. A similar argument shows that $\sup(u-v) \leq 0$. Thus $u \equiv v$.

THEOREM 5. *There exists K with the following properties*:

(i) *K tends to a negative constant near ∂B.*

(ii) *There are two distinct complete metrics $g_1 = e^{2u}h$ and $g_2 = e^{2v}h$ uniformly and conformally equivalent to h and both having curvature K.*

PROOF. Let $\phi \in C^\infty(B)$ satisfy

(i) $\phi \neq 0$ in B,

(ii) $\phi \equiv 1$ in a neighborhood of the origin,

(iii) $\phi = (1-r^2)^{1/2+\sqrt{3}/2} = (1-r^2)^\alpha$ near $r = 1$.

Define $f = \Delta\phi/(e^{2\phi} - 1)$. Then f is a smooth function which is identically 0 near the origin. We have seen in §1 that

$$\Delta(1-r^2)^\alpha = (1-r^2)^\alpha(4\alpha^2r^2 - 4\alpha).$$

Thus, near $r = 1$, f is approximately equal to $(1-r^2)^\alpha(4\alpha^2r^2 - 4\alpha)/2(1-r^2)^\alpha$. By the choice of α, we see that $f = 1 + \mathcal{O}(1-r^2)$.

It is thus possible to find a bounded solution, $v \in C^\infty(B)$, of the equation $\Delta v = f - 1$. We define $K = -(\Delta v + 1)e^{-2v}$ and $u = v + \phi$.

Then $\Delta v = -1 - Ke^{2v}$ by definition and

$$\Delta u = (f-1) + f(e^{2\phi} - 1) = (\Delta v + 1)e^{2u}e^{-2v} - 1 = -1 - Ke^{2u}.$$

It is easy to check that K tends to a negative constant near ∂B. □

NOTE ADDED IN PROOF. Because of the normalization involved in our choice of hyperbolic metric h, equation (*) should read: $\Delta_h u = -4 - K_g e^{2u}$. This will not affect any of the results, but it will lead to corresponding changes throughout the paper.

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