PARTITIONS AND DIAMOND
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ABSTRACT. We restate the diamond principle in terms of partitions, and we show that a weakening of diamond follows from the generalized continuum hypothesis.

For the duration of this paper $\kappa$ will denote a fixed regular uncountable cardinal. Our set-theoretic notation is that of [4].

For every ordinal $\alpha \leq \kappa$, let $(\kappa)^\alpha$ denote the collection of those sequences $X(\nu)$, $\nu < \alpha$, of pairwise disjoint nonempty subsets of $\kappa$ such that $\bigcup_{\nu < \alpha} X(\nu) = \kappa$ and $\bigcap X(\nu) < \bigcap X(\mu)$ whenever $\nu < \mu$. We recall that an ideal $I$ over $\kappa$ is $\kappa$-saturated if for every $X \in (\kappa)^\kappa$, there exists $\alpha < \kappa$ such that $X(\alpha) \in I$.

The following is an easy restatement of what is widely known as Ketonen’s lemma (for a proof see Lemma 33.3 of [4]).

**Proposition 1.** $\kappa < \kappa = \kappa$ iff for every non-$\kappa$-saturated ideal $I$ over $\kappa$, there exists a family $X_\alpha \in (\kappa)^\kappa$, $\alpha < 2^\kappa$, such that for every $A \in [2^\kappa]^\kappa$ and every $h \in \kappa^A$, $\bigcap_{\alpha \in A} X_\alpha(h(\alpha)) \not\in I$.

Ketonen’s lemma has applications in the theory of ultrafilters. One corollary states that if $\kappa$ is $2^\kappa$-compact and $H$ is a $\kappa$-complete filter over $\kappa$ such that its dual ideal is not $\kappa$-saturated, then there are $2^{2^\kappa}$ $\kappa$-complete ultrafilters over $\kappa$ extending $H$. We omit the proof of the following, as it closely follows the proof of the related Theorem 3.2 of [6].

**Corollary 2.** Assume $\kappa < \kappa = \kappa$, and let $H$ be a nontrivial filter over $\kappa$ such that its dual ideal is not $\kappa$-saturated. Then $H$ can be extended to an ultrafilter $K$ over $\kappa$ with the following property: If $p : [\kappa]^\kappa \rightarrow K$ is a function such that $p(u) \supseteq p(w)$ whenever $u \subseteq w$, then there exists a function $q : [\kappa]^\kappa \rightarrow K$ such that $q(u) \subseteq p(u)$ for every $u$, and that for every family $u_\alpha$, $\alpha < \kappa$, of members of $[\kappa]^\kappa$, $q(\bigcup_{\alpha < \gamma} u_\alpha) = \bigcap_{\alpha < \gamma} q(u_\alpha)$.

It is easy to modify the proof of Ketonen’s lemma so as to obtain the following.

**Proposition 3.** $\kappa < \kappa = \kappa$ iff there exists a family $X_\alpha \in (\kappa)^\kappa$, $\alpha < 2^\kappa$, such that for every $A \in [2^\kappa]^\kappa$ and every $h \in \kappa^A$, there is an $E \in [\kappa]^\kappa$ with the property that $|E - X_\alpha(h(\alpha))| < \kappa$ for every $\alpha \in A$.

**Proof.** Assume $\kappa < \kappa = \kappa$. Choose a family $B_\alpha$, $\alpha < 2^\kappa$, of members of $[\kappa]^\kappa$ such that $|B_\alpha \cap B_\beta| < \kappa$ whenever $\alpha \neq \beta$. For each $\alpha < 2^\kappa$, let $f_\alpha$ be a mapping of $B_\alpha$ onto $\kappa$ such that $|\{\gamma \in B_\alpha : f_\alpha(\gamma) = \beta\}| = \kappa$ for every $\beta < \kappa$. Let $s_\gamma$, $\gamma < \kappa$, enumerate $[\kappa]^\kappa$. For each $\alpha < 2^\kappa$, define $g_\alpha : \kappa \rightarrow \kappa$ by letting $g_\alpha(\gamma) = f_\alpha(\beta)$ when...
s_\gamma \cap B_\alpha = \{\beta\}, \text{ and } g_\alpha(\gamma) = 0 \text{ if } |s_\gamma \cap B_\alpha| \neq 1. \text{ Also define } X_\alpha \in (\kappa)^\kappa \text{ by letting } 
abla_\gamma, \delta \text{ lie in the same piece of } X_\alpha \text{ iff } g_\alpha(\gamma) = g_\alpha(\delta). \text{ Fix } A \subseteq [2^\kappa]^\kappa \text{ and } h \in \kappa^A. \text{ Let } k \in \kappa^A \text{ be such that } \gamma \in X_\alpha(h(\alpha)) \text{ iff } g_\alpha(\gamma) = k(\alpha). \text{ Letting } a_\beta, \beta < \kappa, \text{ enumerate } A, \text{ define } t \in \kappa^\kappa \text{ as follows: Given } \gamma < \kappa, \forall \beta < \gamma \text{ choose } \delta_\beta \in B_{a_\beta} \text{ such that } \delta_\beta \notin B_{a_\alpha} \text{ for } \alpha \in \gamma \setminus \{\beta\}, \text{ and } f_{a_\beta}(\delta_\beta) = k(a_\beta). \text{ Then pick } t(\gamma) \text{ so that } s_t(\gamma) = \{\delta_\beta: \beta < \gamma\}. \text{ Finally put } E = \{t(\gamma): \gamma < \kappa\}. \text{ Clearly, } |E - X_\alpha(h(\alpha))| < \kappa \text{ for every } \alpha \in A.

Now for the other direction: Let \( X_\alpha \in (\kappa)^\kappa, \alpha < 2^\kappa, \) as in the statement of the proposition. It suffices to observe that for every cardinal \( \mu < \kappa, \) the family \( \bigcap_{\alpha < \mu} X_\alpha(h(\alpha)), h \in \kappa^\mu, \) consists of pairwise disjoint nonempty subsets of \( \kappa. \)

As formulated by Jensen, \( \Diamond_\kappa \) asserts the existence of a family \( S_\alpha, \alpha < \kappa, \) with each \( S_\alpha \subseteq \alpha \) such that for any \( A \subseteq \kappa, \) the set \( \{\alpha: A \cap \alpha = S_\alpha\} \) is stationary in \( \kappa. \) It is well known that \( \Diamond_\kappa \) is equivalent to the existence of functions \( g_\alpha: \alpha \to \alpha, \alpha < \kappa, \) such that for any \( f: \kappa \to \kappa, \) the set \( \{\alpha: f \upharpoonright \alpha = g_\alpha\} \) is stationary. Also, Devlin \[2\] showed that \( \Diamond_\kappa \) holds iff there are \( S_\alpha, \alpha < \kappa, \) with each \( S_\alpha \subseteq \alpha \) such that for any \( A \subseteq \kappa, \) one can find \( \alpha \geq \omega \) with \( A \cap \alpha = S_\alpha. \) Jensen \[5\] proved that \( \Diamond_\kappa \) follows from \( V = L. \)

**PROPOSITION 4.** The following are equivalent:

(i) \( \Diamond_\kappa. \)

(ii) There exists a family \( X_\alpha \in (\kappa)^\kappa, \alpha < \kappa, \) such that the diagonal intersection \( \Delta\{X_\alpha(h(\alpha)): \alpha < \kappa\} \) is stationary for every \( h \in \kappa^\kappa. \)

(iii) There exists a family \( X_\alpha \in (\kappa)^\kappa, \alpha < \kappa, \) such that for every \( h \in \kappa^\kappa, \) there is a stationary \( E \in [\kappa]^\kappa \) with the property that \( |E - X_\alpha(h(\alpha))| < \kappa \) for every \( \alpha < \kappa. \)

(iv) There exists a family \( Z_\alpha \in (\kappa)^2, \alpha < \kappa, \) such that for every \( h \in 2^\kappa, \) there is a \( \beta \geq \omega \) with \( \beta \in \bigcap_{\alpha < \beta} Z_\alpha(h(\alpha)). \)

**PROOF.** (i)\( \rightarrow \) (ii): Choose functions \( g_\alpha: \alpha \to \alpha, \alpha < \kappa, \) such that for any \( f: \kappa \to \kappa, \) the set \( \{\alpha: f \upharpoonright \alpha = g_\alpha\} \) is stationary. For each \( \alpha < \kappa, \) put \( A_\alpha^0 = (\alpha + 1) \cup \{\beta > \alpha: g_\beta(\alpha) = 0\}, \) and for each \( \gamma > 0, A_\gamma^0 = \{\beta > \alpha: g_\beta(\alpha) = \gamma\}. \) Let \( X_\alpha \) denote the partition of \( \kappa \) into the pieces \( A_\gamma^0, \gamma < \kappa. \) Fix \( h \in \kappa^\kappa, \) and let \( f \in \kappa^\kappa \) be such that \( X_\alpha(h(\alpha)) = A_f^\alpha(\alpha) \) for every \( \alpha < \kappa. \) Set \( E = \{\alpha: f \upharpoonright \alpha = g_\alpha\}. \) Now it is easy to see that \( \beta \in E \) iff \( \beta \in X_\alpha(h(\alpha)) \) for every \( \alpha < \beta. \) Thus \( \Delta\{X_\alpha(h(\alpha)): \alpha < \kappa\} \) is stationary.

(ii)\( \rightarrow \) (iii) is trivial.

(iii)\( \rightarrow \) (iv): Pick a family \( Z_\alpha \in (\kappa)^2, \alpha < \kappa, \) such that for each \( h \in 2^\kappa, \) there is a stationary \( E_\hbar \subseteq [\kappa]^\kappa \) with the property that \( |E_\hbar - Z_\alpha(h(\alpha))| < \kappa \) for every \( \alpha < \kappa. \) Fix \( h \in 2^\kappa. \) In case \( E - \beta \subseteq \bigcap_{\alpha < \kappa} Z_\alpha(h(\alpha)) \) for some \( \beta, \) there is nothing to prove, so assume otherwise. We define a function \( k: E_\hbar \to \kappa \) by letting \( k(\beta) \) be the least \( \alpha \) with \( E - \beta \not\subseteq Z_\alpha(h(\alpha)). \) If the set \( \{\beta \in E_\hbar: k(\beta) < \beta\} \) were stationary, then \( k \) would be constant on some \( H \in [E]^\kappa, \) a contradiction. Hence, \( \{\beta \in E_\hbar: k(\beta) \geq \beta\} \) is stationary.

(iv)\( \rightarrow \) (i): Pick a family \( Z_\alpha \in (\kappa)^2, \alpha < \kappa, \) as in (iv). For each \( \beta < \kappa, \) put \( S_\beta = \{\alpha < \beta: \beta \in Z_\alpha(1)\}. \) Given \( A \subseteq \kappa, \) define \( h: \kappa \to 2 \) by letting \( h(\alpha) = 1 \) iff \( \alpha \in A. \) Choose \( \beta \geq \omega \) with \( \beta \in \bigcap_{\alpha < \beta} Z_\alpha(h(\alpha)). \) Then for each \( \alpha < \beta, \)

\[ \alpha \in S_\beta \leftrightarrow \beta \in Z_\alpha(1) \leftrightarrow h(\alpha) = 1 \leftrightarrow \alpha \in A. \]

Thus \( S_\beta = A \cap \beta. \)
Let $\lambda \geq \kappa$ be a fixed uncountable cardinal. The combinatorial principle $\Diamond_{\kappa, \lambda}$ which was introduced by Jech in [3] asserts the existence of a family $S_P, P \in [\lambda]^{<\kappa}$, with each $S_P \subseteq P$ such that for any $A \subseteq \lambda$, the set $\{P: A \cap P = S_P\}$ is stationary in $[\lambda]^{<\kappa}$. It is readily verified that $\Diamond_{\kappa, \lambda}$ holds iff there are partitions $Z_\alpha, \alpha < \lambda$, of $[\lambda]^{<\kappa}$ into two pieces $Z_\alpha(0)$ and $Z_\alpha(1)$ such that the diagonal intersection $\Delta\{Z_\alpha(h(\alpha)) : \alpha < \lambda\}$ is stationary in $[\lambda]^{<\kappa}$ for every $h \in 2^\lambda$.

REFERENCES


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