

PARTITIONS AND DIAMOND

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ABSTRACT. We restate the diamond principle in terms of partitions, and we show that a weakening of diamond follows from the generalized continuum hypothesis.

For the duration of this paper κ will denote a fixed regular uncountable cardinal. Our set-theoretic notation is that of [4].

For every ordinal $\alpha \leq \kappa$, let $(\kappa)^\alpha$ denote the collection of those sequences $X(\nu)$, $\nu < \alpha$, of pairwise disjoint nonempty subsets of κ such that $\bigcup_{\nu < \alpha} X(\nu) = \kappa$ and $\bigcap X(\nu) < \bigcap X(\mu)$ whenever $\nu < \mu$. We recall that an ideal I over κ is κ -saturated if for every $X \in (\kappa)^\kappa$, there exists $\alpha < \kappa$ such that $X(\alpha) \in I$.

The following is an easy restatement of what is widely known as Ketonen's lemma (for a proof see Lemma 33.3 of [4]).

PROPOSITION 1. $\kappa^{<\kappa} = \kappa$ iff for every non- κ -saturated ideal I over κ , there exists a family $X_\alpha \in (\kappa)^\kappa$, $\alpha < 2^\kappa$, such that for every $A \in [2^\kappa]^{<\kappa}$ and every $h \in \kappa^A$, $\bigcap_{\alpha \in A} X_\alpha(h(\alpha)) \notin I$.

Ketonen's lemma has applications in the theory of ultrafilters. One corollary states that if κ is 2^κ -compact and H is a κ -complete filter over κ such that its dual ideal is not κ -saturated, then there are 2^{2^κ} κ -complete ultrafilters over κ extending H . We omit the proof of the following, as it closely follows the proof of the related Theorem 3.2 of [6].

COROLLARY 2. Assume $\kappa^{<\kappa} = \kappa$, and let H be a nontrivial filter over κ such that its dual ideal is not κ -saturated. Then H can be extended to an ultrafilter K over κ with the following property: If $p: [\kappa]^{<\kappa} \rightarrow K$ is a function such that $p(u) \supseteq p(w)$ whenever $u \subseteq w$, then there exists a function $q: [\kappa]^{<\kappa} \rightarrow K$ such that $q(u) \subseteq p(u)$ for every u , and that for every family u_α , $\alpha < \gamma < \kappa$, of members of $[\kappa]^{<\kappa}$, $q(\bigcup_{\alpha < \gamma} u_\alpha) = \bigcap_{\alpha < \gamma} q(u_\alpha)$.

It is easy to modify the proof of Ketonen's lemma so as to obtain the following.

PROPOSITION 3. $\kappa^{<\kappa} = \kappa$ iff there exists a family $X_\alpha \in (\kappa)^\kappa$, $\alpha < 2^\kappa$, such that for every $A \in [2^\kappa]^\kappa$ and every $h \in \kappa^A$, there is an $E \in [\kappa]^\kappa$ with the property that $|E - X_\alpha(h(\alpha))| < \kappa$ for every $\alpha \in A$.

PROOF. Assume $\kappa^{<\kappa} = \kappa$. Choose a family B_α , $\alpha < 2^\kappa$, of members of $[\kappa]^\kappa$ such that $|B_\alpha \cap B_\beta| < \kappa$ whenever $\alpha \neq \beta$. For each $\alpha < 2^\kappa$, let f_α be a mapping of B_α onto κ such that $|\{\gamma \in B_\alpha: f_\alpha(\gamma) = \beta\}| = \kappa$ for every $\beta < \kappa$. Let s_γ , $\gamma < \kappa$, enumerate $[\kappa]^{<\kappa}$. For each $\alpha < 2^\kappa$, define $g_\alpha: \kappa \rightarrow \kappa$ by letting $g_\alpha(\gamma) = f_\alpha(\beta)$ when

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$s_\gamma \cap B_\alpha = \{\beta\}$, and $g_\alpha(\gamma) = 0$ if $|s_\gamma \cap B_\alpha| \neq 1$. Also define $X_\alpha \in (\kappa)^\kappa$ by letting γ, δ lie in the same piece of X_α iff $g_\alpha(\gamma) = g_\alpha(\delta)$. Fix $A \in [2^\kappa]^\kappa$ and $h \in \kappa^A$. Let $k \in \kappa^A$ be such that $\gamma \in X_\alpha(h(\alpha))$ iff $g_\alpha(\gamma) = k(\alpha)$. Letting $a_\beta, \beta < \kappa$, enumerate A , define $t \in \kappa^\kappa$ as follows: Given $\gamma < \kappa$, for each $\beta < \gamma$ choose $\delta_\beta \in B_{a_\beta}$ such that $\delta_\beta \notin B_{a_\alpha}$ for $\alpha \in \gamma - \{\beta\}$, and $f_{a_\beta}(\delta_\beta) = k(a_\beta)$. Then pick $t(\gamma)$ so that $s_{t(\gamma)} = \{\delta_\beta: \beta < \gamma\}$. Finally put $E = \{t(\gamma): \gamma < \kappa\}$. Clearly, $|E - X_\alpha(h(\alpha))| < \kappa$ for every $\alpha \in A$.

Now for the other direction: Let $X_\alpha \in (\kappa)^\kappa, \alpha < 2^\kappa$, be as in the statement of the proposition. It suffices to observe that for every cardinal $\mu < \kappa$, the family $\bigcap_{\alpha < \mu} X_\alpha(h(\alpha)), h \in \kappa^\mu$, consists of pairwise disjoint nonempty subsets of κ .

As formulated by Jensen, \diamond_κ asserts the existence of a family $S_\alpha, \alpha < \kappa$, with each $S_\alpha \subseteq \alpha$ such that for any $A \subseteq \kappa$, the set $\{\alpha: A \cap \alpha = S_\alpha\}$ is stationary in κ . It is well known that \diamond_κ is equivalent to the existence of functions $g_\alpha: \alpha \rightarrow \alpha, \alpha < \kappa$, such that for any $f: \kappa \rightarrow \kappa$, the set $\{\alpha: f \upharpoonright \alpha = g_\alpha\}$ is stationary. Also, Devlin [2] showed that \diamond_κ holds iff there are $S_\alpha, \alpha < \kappa$, with each $S_\alpha \subseteq \alpha$ such that for any $A \subseteq \kappa$, one can find $\alpha \geq \omega$ with $A \cap \alpha = S_\alpha$. Jensen [5] proved that \diamond_κ follows from $V = L$.

PROPOSITION 4. *The following are equivalent:*

- (i) \diamond_κ .
- (ii) *There exists a family $X_\alpha \in (\kappa)^\kappa, \alpha < \kappa$, such that the diagonal intersection $\Delta\{X_\alpha(h(\alpha)): \alpha < \kappa\}$ is stationary for every $h \in \kappa^\kappa$.*
- (iii) *There exists a family $X_\alpha \in (\kappa)^\kappa, \alpha < \kappa$, such that for every $h \in \kappa^\kappa$, there is a stationary $E \in [\kappa]^\kappa$ with the property that $|E - X_\alpha(h(\alpha))| < \kappa$ for every $\alpha < \kappa$.*
- (iv) *There exists a family $Z_\alpha \in (\kappa)^2, \alpha < \kappa$, such that for every $h \in 2^\kappa$, there is a $\beta \geq \omega$ with $\beta \in \bigcap_{\alpha < \beta} Z_\alpha(h(\alpha))$.*

PROOF. (i)→(ii): Choose functions $g_\alpha: \alpha \rightarrow \alpha, \alpha < \kappa$, such that for any $f: \kappa \rightarrow \kappa$, the set $\{\alpha: f \upharpoonright \alpha = g_\alpha\}$ is stationary. For each $\alpha < \kappa$, put $A_0^\alpha = (\alpha + 1) \cup \{\beta > \alpha: g_\beta(\alpha) = 0\}$, and for each $\gamma > 0, A_\gamma^\alpha = \{\beta > \alpha: g_\beta(\alpha) = \gamma\}$. Let X_α denote the partition of κ into the pieces $A_\gamma^\alpha, \gamma < \kappa$. Fix $h \in \kappa^\kappa$, and let $f \in \kappa^\kappa$ be such that $X_\alpha(h(\alpha)) = A_{f(\alpha)}^\alpha$ for every $\alpha < \kappa$. Set $E = \{\alpha: f \upharpoonright \alpha = g_\alpha\}$. Now it is easy to see that $\beta \in E$ iff $\beta \in X_\alpha(h(\alpha))$ for every $\alpha < \beta$. Thus $\Delta\{X_\alpha(h(\alpha)): \alpha < \kappa\}$ is stationary.

(ii)→(iii) is trivial.

(iii)→(iv): Pick a family $Z_\alpha \in (\kappa)^2, \alpha < \kappa$, such that for each $h \in 2^\kappa$, there is a stationary $E_h \in [\kappa]^\kappa$ with the property that $|E_h - Z_\alpha(h(\alpha))| < \kappa$ for every $\alpha < \kappa$. Fix $h \in 2^\kappa$. In case $E - \beta \subseteq \bigcap_{\alpha < \kappa} Z_\alpha(h(\alpha))$ for some β , there is nothing to prove, so assume otherwise. We define a function $k: E_h \rightarrow \kappa$ by letting $k(\beta)$ be the least α with $E - \beta \not\subseteq Z_\alpha(h(\alpha))$. If the set $\{\beta \in E_h: k(\beta) < \beta\}$ were stationary, then k would be constant on some $H \in [E]^\kappa$, a contradiction. Hence, $\{\beta \in E_h: k(\beta) \geq \beta\}$ is stationary.

(iv)→(i): Pick a family $Z_\alpha \in (\kappa)^2, \alpha < \kappa$, as in (iv). For each $\beta < \kappa$, put $S_\beta = \{\alpha < \beta: \beta \in Z_\alpha(1)\}$. Given $A \subseteq \kappa$, define $h: \kappa \rightarrow 2$ by letting $h(\alpha) = 1$ iff $\alpha \in A$. Choose $\beta \geq \omega$ with $\beta \in \bigcap_{\alpha < \beta} Z_\alpha(h(\alpha))$. Then for each $\alpha < \beta$,

$$\alpha \in S_\beta \leftrightarrow \beta \in Z_\alpha(1) \leftrightarrow h(\alpha) = 1 \leftrightarrow \alpha \in A.$$

Thus $S_\beta = A \cap \beta$.

Let $\lambda \geq \kappa$ be a fixed uncountable cardinal. The combinatorial principle $\diamond_{\kappa, \lambda}$ which was introduced by Jech in [3] asserts the existence of a family S_P , $P \in [\lambda]^{<\kappa}$, with each $S_P \subseteq P$ such that for any $A \subseteq \lambda$, the set $\{P: A \cap P = S_P\}$ is stationary in $[\lambda]^{<\kappa}$. It is readily verified that $\diamond_{\kappa, \lambda}$ holds iff there are partitions Z_α , $\alpha < \lambda$, of $[\lambda]^{<\kappa}$ into two pieces $Z_\alpha(0)$ and $Z_\alpha(1)$ such that the diagonal intersection $\Delta\{Z_\alpha(h(\alpha)): \alpha < \lambda\}$ is stationary in $[\lambda]^{<\kappa}$ for every $h \in 2^\lambda$.

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