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ANOTHER COUNTEREXAMPLE IN ANR THEORY

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ABSTRACT. We answer an old question due to Kuratowski by constructing a (separable metric) space X having the following properties: (1) X is not an ANR, and (2) for every space Y and for every compact $A \subseteq Y$, every continuous map $f: A \to X$ can be continuously extended to a map $\bar{f}: Y \to X$.

1. Introduction. All spaces under discussion are separable metric, and for all undefined notions see [**1**, **3** and **5**].

A space X has the compact extension property (abbreviated CEP) if for every space Y and for every compact $A \subseteq Y$, every continuous function $f: A \to X$ can be extended to a continuous function $\bar{f}: Y \to X$. It is easy to show that $\mathrm{AR} \Rightarrow \mathrm{CEP} \Rightarrow C^\infty$ and LC^∞. For details, see [**3** and **9**].

Clearly, for compact X we have $X \in \mathrm{AR}$ if and only if $X \in \mathrm{CEP}$. In addition, if X is finite-dimensional, then $X \in \mathrm{AR}$ if and only if $X \in C^\infty$ and $X \in LC^\infty$; see [**3**, **5**]. It is known that the property of being C^∞ and LC^∞ does not imply CEP since Borsuk [**3**] constructed an example of a contractible and locally contractible compactum which is not an AR. The question naturally arises whether CEP implies AR. This was asked by Kuratowski [**9**] in 1951. The aim of this note is to answer this question in the negative by constructing a space X having the following properties:

(1) X is not an ANR.

(2) For every space Y, for every analytic closed subspace $A \subseteq Y$, every continuous function $f: A \to Y$ can be extended to a continuous function $\bar{f}: Y \to X$.

Recall that a space is analytic if it is a continuous image of the space of irrational numbers. Clearly, every compact space is analytic. It is even true that every topologically complete space is analytic [**10**]. As in many counterexamples in ANR theory [**4**, **11**], the Taylor example [**12**] is an essential ingredient in our example.

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2. The example. By Taylor [**12**] there exists a compact space T and a cell-like mapping $f: T \to M$, where M is homeomorphic to the Hilbert cube Q, which is not a shape equivalence. We assume that T is a Z-set in Q. Let $Z = Q \cup_f M$, and let $\pi: Q \to Z$ be the adjunction projection. It is well known that Z is not an ANR since π is not a hereditary shape equivalence. Clearly, π is cell-like. Put $Y = Z \times I$, where I denotes the interval $[0,1]$. Since $M \times I$ is compact, we can write it as the disjoint union of two sets, say A and B, which both do not contain any Cantor set [**10**, p. 259]. Our example is

$$X = (\pi(Q \setminus T) \times I) \cup A.$$

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In the remaining part of this section we shall prove that X is as required.

2.1. LEMMA. *If $F \subseteq A$ is countable, then $X(F) = (\pi(Q \setminus T) \times I) \cup F$ is an AR.*

PROOF. Define $\rho: Q \times I \to Y$ by $\rho = \pi \times \mathrm{id}_I$. Then ρ is clearly cell-like. Put $S(F) = \rho^{-1}(X(F))$ and $\bar{\rho} = \rho|S(F)$. Then $\bar{\rho}: S(F) \to X(F)$ is cell-like. Since $(Q \setminus T) \times I \subseteq S(F)$ and $T \times I$ is a Z-set in the Hilbert cube $Q \times I$, the proof of [**2**, Theorem 3.1] shows that $S(F)$ is an AR. Consequently, $\bar{\rho}: S(F) \to X(F)$ is a cell-like mapping which is defined on an AR, while, moreover, its nondegeneracy set is contained in the countable subset F of $X(F)$. This implies that $\bar{\rho}$ is a hereditary shape equivalence and in turn that $X(F)$ is an AR [**8**] (see also [**1**]). □

2.2. LEMMA. *For every space E and every analytic closed $F \subseteq E$, every continuous function $f: F \to X$ can be extended to a continuous function $\bar{f}: E \to X$.*

PROOF. Let E be a space and let $F \subseteq E$ be closed and analytic. In addition, let $f: F \to X$ be continuous. Then $f(F)$ is analytic, and since A is closed in X, $f(F) \cap A$ is closed in $f(F)$, whence $H = f(F) \cap A$ is analytic as well [**10**]. Since every uncountable analytic space contains a Cantor set [**10**] by the special choice of A it follows that H is countable. From Lemma 2.1 we therefore conclude that $f(F)$ is contained in the AR $X(H)$. Consequently, f can be extended to a continuous function $\bar{f}: E \to X(H) \subseteq X$. □

We shall now prove that X is not an ANR.

2.3. LEMMA. *X is not an ANR.*

PROOF. To the contrary, assume that X is an ANR. By Lemma 2.2 every continuous function $f: S^n \to X$, where $n \in \{0, 1, 2, \ldots\}$, extends to a continuous function $\bar{f}: B^{n+1} \to X$. Consequently, X is C^∞ and an ANR, so X is in fact an AR [**5**, III, 7.3]. Let N be a homeomorph of Q containing $Y = Z \times I$. Define $T = N \setminus B$. Observe that X is a closed subset of T. Since X is an AR, there is a retraction $r: T \to X \subseteq Y$. Since Y is compact, and hence topologically complete, there is a G_δ-set S of N containing T such that r can be extended to a continuous function $\bar{r}: S \to Y$ (this is well known and easy to prove). Observe that $N \setminus S$ is an F_σ-subset of N which is contained in B. Since B contains no Cantor sets, it follows that $N \setminus S$ is countable. From [**2**, Theorem 3.1] we conclude that S is an AR. Since $N \setminus S$ is countable, we find that $B \setminus S$ is countable, and, consequently, there exists a point $t \in I$ such that $(Z \times \{t\}) \cap (N \setminus S) = \emptyset$. Put $D = Z \times \{t\}$ and define $\xi: Y \to D$ by $\xi(x, s) = (x, t)$. Since $D \subseteq S$ and $r|(D \setminus B)$ is the identity, we also find that $\xi \circ \bar{r}$ is the identity on $D \setminus B$. Now since, clearly, $D \setminus B$ is dense in D, $\xi \circ \bar{r}$ is a retraction from S onto D. Since, as was remarked above, S is an AR and D, being homeomorphic to Z, is not, we have derived the desired contradiction. □

Question. Let X be an absolute Borel set with the compact extension property. Is X an AR?

REMARK. A linear space E is admissible if every compact subset of E can be pushed by arbitrarily small maps into finite-dimensional linear subspaces of E. Every locally convex space is admissible, but there exist nonlocally convex spaces which are also admissible, e.g. l^p for $p < 1$ [**7**]. It is known that every admissible topologically complete linear space has the compact extension property [**6**]. Apparently, it is still unknown whether *every* linear space is admissible.

ADDED IN PROOF. I recently showed that if there exists a cell-like dimension raising mapping between compact spaces then there exists a topologically complete non-ANR with the CEP.

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