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PSEUDOFREE REPRESENTATIONS AND 2-PSEUDOFREE ACTIONS ON SPHERES

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ABSTRACT. We characterize 1-pseudofree real representations of finite groups. We apply this to show that the representations at fixed points of a 2-pseudofree smooth action of a finite group on a sphere of dimension ≥ 5 are topologically equivalent. Moreover with one possible exception, the sphere is G-homeomorphic to a linear representation sphere.

1. Introduction. Let a finite group G act smoothly on a manifold M. Then G acts linearly on the tangent space of any fixed point $a \in M^G$. Call this G-module the representation of G at a. Cappell and Shaneson have conjectured: If M is a (mod 2) homology sphere and M^H is finite or connected for every cyclic subgroup H of G, then the representations of G at any two fixed points are topologically equivalent.

In [**6**] Illman considered the problem in the situation where M is a homotopy sphere and one of the local representations is 2-pseudofree. A real G-module V is called n-pseudofree if G acts freely on V outside a set of dimension at most n.

THEOREM 1. *Let G be a finite group acting smoothly on a homotopy sphere Σ^m, $m \geq 5$, such that there is a fixed point $a \in \Sigma^G$ at which the local representation ρ_a of G is 2-pseudofree. Then*

(i) *the representations of G at any two fixed points are topologically equivalent,*

(ii) *if there are at least two fixed points, then Σ^m is G-homeomorphic to the linear representation sphere $S^m(\rho_a \oplus 1)$,*

(iii) *if there is only one fixed point, then $G = A_5$ and $\Sigma^m = S^6$.*

We have not been able to decide whether the exceptional case (iii) can really occur. Combining Illman's approach to Theorem 1 with the group theory of §2 of this paper we prove Theorem 1 in the general case. Illman [**6**] dealt with the case where G is solvable or arbitrary but ρ_a is 1-pseudofree.

Theorem 1 is false without some restriction on ρ_a, such as 2-pseudofreeness. Indeed, Petrie gave in [**11**, Theorem B] examples of free smooth actions of odd order abelian groups G on a homotopy sphere with precisely two fixed points a and b such that ρ_a and ρ_b are not linearly equivalent. Then the representations ρ_a and ρ_b must be even topologically nonequivalent, since topologically equivalent representations of groups of odd order are linearly equivalent by the celebrated results of Hsiang and Pardon [**3**, **4**] and Madsen and Rothenberg [**9**].

Our work is mostly group theoretic and depends on a study of 1- and 2-pseudofree representations. We show that except for a few low-dimensional examples the 1-pseudofree representations reduce to the free representations. More precisely, let

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D_{2m} be the dihedral group of order $2m$ and let T, O and I be the polyhedral groups of order 12, 24 and 60 (isomorphic to A_4, S_4 and A_5, see [**14**, 2.6]).

THEOREM 2. *Let $\rho\colon G \to O(n)$ be a 1-pseudofree representation which is not free, with $n \geq 2$. Then one of the following conditions holds*:

(a) $n = 2,\ G = D_{2m}$,

(b) $n = 3,\ G = D_{2m},\ T, O$ *or* I, *and* $\rho(G) \subset SO(3)$,

(c) *there is a free representation* $\psi\colon G \to O(n-1)$ *and a homomorphism* $\chi\colon G \to O(1)$ *such that* $\rho = \psi + \chi$.

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2. Characterization of 1-pseudofree representations. Let G be a finite group. An orthogonal representation of G on a real vector space V is *n-pseudofree* if $\dim V^H \leq n$ for every subgroup $H \leq G$. An 0-pseudofree representation is thus what is usually called *free*, and for brevity we call 1-pseudofree representations *pseudofree.*

A representation $\rho\colon G \to O(V)$ is pseudofree (resp. free) if and only if for each $g \in G,\ g \neq e$, the fixed point space V^g has dimension at most 1 (resp. 0) or equivalently if $\rho(g)$ has 1 as an eigenvalue at most once (resp. never). If $\dim V = 1$, then any representation of G on V is pseudofree but if $\dim V \geq 2$, then a pseudofree representation on V must be faithful since $V^g = V$ for $g \in \operatorname{Ker} \rho$.

Any subrepresentation of a pseudofree representation is pseudofree. Hence a pseudofree representation is a direct sum of irreducible pseudofree representations. Let $\rho\colon G \to O(n)$ be a real irreducible representation. Then either (i) ρ is irreducible as a complex representation, i.e. the complex representation $G \xrightarrow{\rho} O(n) \subset U(n)$ is irreducible, or (ii) $n = 2m$ and ρ is obtained from a unitary representation of G on $\mathbf{C}^m$ by the identification $\mathbf{C}^m = \mathbf{R}^{2m}$ [**12**, §13.2].

In case (ii) all real eigenvalues of $\rho(g)$ appear twice, hence ρ is free if it is pseudofree. For groups of odd order, every nontrivial irreducible ρ has type (ii). Thus if G has odd order and $\rho\colon G \to O(m)$ is a pseudofree representation, then ρ is free when m is even and $\rho = \psi + 1$, where ψ is free when m is odd.

It turns out that the dihedral groups

$$D_{2m} = \langle A, B | A^m = 1,\ B^2 = 1,\ BAB^{-1} = A^{-1} \rangle$$

of order $2m,\ m \geq 2$, play a crucial role in 1- and 2-pseudofree representations. Call an element of a group an involution if it has order 2. Then D_{2m} is generated by the two involutions $x = AB$ and $y = B$, and conversely two different involutions x and y in a finite group G generate a dihedral group $\langle x, y \rangle$ (send xy to A and y to B).

LEMMA 2.1. *If G is a dihedral group and $\rho\colon G \to O(V)$ is an n-pseudofree representation, then* $\dim V \leq 3n$. *If moreover* $\dim V \geq 3n - 1$, *then* $\rho(G) \subset SO(V)$ *if and only if* $\dim V = 3n$, *and in this case* $V^G = \{0\}$.

PROOF. Choose two different involutions x and y generating G. Let $V_1 = V^x,\ V_2 = V^y$ and denote by U_i the orthogonal complement of V_i in $V,\ i = 1, 2$. Since G acts orthogonally, U_1 is the (-1)-eigenspace of x and U_2 is the (-1)-eigenspace of y. Then xy fixes each element of $U_1 \cap U_2 = (V_1 + V_2)^{\perp}$. Also $xy \neq e$

since $x \neq y = y^{-1}$. The assumption that ρ is n-pseudofree implies

$$(*) \qquad \begin{aligned} n &\geq \dim(U_1 \cap U_2) = \dim V - \dim(V_1 + V_2) \\ &\geq \dim V - \dim V_1 - \dim V_2 \geq \dim V - 2n, \end{aligned}$$

hence $\dim V \leq 3n$.

As x and y generate G, $\rho(G) \subset SO(V)$ if and only if both $\rho(x)$ and $\rho(y)$ have determinant 1. These determinants are $(-1)^{\dim U_1}$ and $(-1)^{\dim U_2}$. If $\dim V = 3n - 1$, then $\dim V_i \leq n$ and $(*)$ imply that $\dim V_i = n$ for at least one i, say $\dim V_1 = n$. Then $\dim U_1 = 2n - 1$ is odd and $\det \rho(x) = -1$. If $\dim V = 3n$, it follows similarly that $\dim V_1 = \dim V_2 = n$, hence $\dim U_1 = \dim U_2 = 2n$ are even and $\det \rho(x) = \det \rho(y) = 1$. Finally in this case $V^G = V^x \cap V^y = V_1 \cap V_2 = \{0\}$. □

The lemma holds clearly for any group G containing a dihedral group, or equivalently containing at least two involutions. Especially, such G can have a pseudofree representation in dimension at most 3, and a 2-pseudofree representation in dimension at most 6. Since a simple nonabelian group G has even order, it contains involutions, and in fact more than one as a unique involution would be central. All representations of G are orientable because $G = G'$. We conclude

COROLLARY 2.2. *Let G be a simple nonabelian group. If $\rho: G \to O(n)$ is a faithful pseudofree representation, then $n = 3$. If $\rho: G \to O(n)$ is a faithful 2-pseudofree representation, then $n = 3, 4$ or 6.* □

The group A_5 has two 3-dimensional 1-pseudofree representations χ_1, χ_2 as the icosahedral group I, hence also 2-pseudofree representations $\chi_i + 1$ in degree 4 and $\chi_1 + \chi_2$, $2\chi_1$ and $2\chi_2$ in degree 6. It is well known that $SO(3)$ does not contain simple groups other than A_5. From the local isomorphism of $SO(4)$ and $SO(3) \times SO(3)$ it follows that $SO(4)$ can contain only A_5 as a finite simple subgroup. Hence the only possible 2-pseudofree simple groups G other than A_5 must occur in degree 6. In §3 we show that $PSL_2(7)$ and A_5 are the only simple groups having a 6-dimensional 2-pseudofree representation. This is probably due to the fact that they are the two simple groups of smallest order (A_5 has order 60 and $PSL_2(7)$ has order 168).

A pseudofree representation $\rho: G \to O(n)$ with $n \geq 2$ is faithful, and G may be considered as a subgroup of $O(n)$. If $n = 2$, then $O(2)$ contains only cyclic or dihedral groups. For $n = 3$ we are reduced to the classification of finite subgroups of $SO(3)$ [**14**, §2.6] by

LEMMA 2.3. *If G is noncyclic and $\rho: G \to O(3)$ is pseudofree, then $\rho(G) \subset SO(3)$.*

PROOF. Let $H = \operatorname{Ker} \det \rho$. Assume $H \neq G$. Then H has index 2 in G and $G = HG_2$, where G_2 is a Sylow 2-subgroup of G. Since the degree of any irreducible representation of G_2 is a power of 2, $\rho | G_2$ must contain a 2-dimensional pseudofree subrepresentation. Hence G_2 is isomorphic to a finite subgroup of $O(2)$, i.e. it is cyclic or dihedral. If $G_2 \cong D_{2^k}$, then $\rho(G_2) \subset SO(3)$ by Lemma 2.1. But then $\rho(G) = \rho(HG_2) \subset SO(3)$, contradicting the assumption $H \neq G$. Thus G_2 is cyclic.

If G contains the scalar matrix $-I$, then $G = H \times \langle -I \rangle$. In this case $G_2 = \langle -I \rangle = Z_2$ and H is a subgroup of $SO(3)$ of odd order, so H and consequently G are cyclic. If $-I \notin G$, then the representation $\rho \det \rho: G \to O(3)$ is injective and its image is contained in $SO(3)$. Hence G is isomorphic to a finite subgroup of $SO(3)$. As G_2 is cyclic, G is cyclic or dihedral. Since $\rho(G) \not\subset SO(3)$, the last case is impossible by Lemma 2.1. Therefore G is cyclic. □

THEOREM 2. *Let $\rho\colon G \to O(n)$ be a 1-pseudofree representation which is not free, with $n \geq 2$. Then one of the following conditions holds:*

(a) $n = 2,\ G = D_{2m}$,

(b) $n = 3,\ G = D_{2m}, T, O, I$ *and* $\rho(G) \subset SO(3)$,

(c) *there is a free representation* $\psi\colon G \to O(n-1)$ *and a homomorphism* $\psi\colon G \to O(1)$ *such that* $\rho = \psi + \chi$.

PROOF. Let $\rho\colon G \to O(n)$ be a pseudofree, nonfree representation with $n \geq 2$. If G is cyclic, we clearly have case (c). Assume G is noncyclic. If $n = 2$, then G is dihedral and we have case (a). If $n = 3$, then we have case (b) by Lemma 2.3 and the knowledge of finite subgroups of $SO(3)$. Assume $n \geq 4$. Then G contains at most one element of order 2 by Lemma 2.1.

Split ρ as a sum $\rho = \bigoplus \rho_i$ of irreducible representations. Then each ρ_i is pseudofree. At most one summand ρ_i can be 1-dimensional. Indeed, if $\rho_i \neq \rho_j$ are 1-dimensional, then $\rho_i \oplus \rho_j\colon G \to O(1) \times O(1) = Z_2 \oplus Z_2$ is pseudofree, hence injective. As G is noncyclic, we must have $G \cong Z_2 \oplus Z_2 = D_4$, contradicting Lemma 2.1 since $n \geq 4$. The claim now follows if we show: Every irreducible pseudofree representation $\rho\colon G \to O(m)$ with $m \geq 2$ of a group having at most one involution is free.

We may assume that ρ is irreducible as a complex representation (otherwise it is automatically free). Especially, G then has even order and contains an involution, hence a unique involution x. As x is central in G, $\rho(x) = -I$ by Schur's lemma.

If the order of G is twice an odd number, then $G = H \times \langle x \rangle$ for some subgroup H of G having odd order by Burnside's theorem [**2**, Theorem 7.4.3]. Then $\rho = \psi \otimes \chi$, where $\psi\colon H \to O(m)$ and $\chi\colon \langle x \rangle \to O(1)$ are irreducible representations. Now $\psi = \rho | H$ is pseudofree and irreducible and H has odd order, hence ψ is free. Then $\chi(x) = -I$ implies that ρ is free.

Assume that 4 divides the order of G. As x is the only involution in G, there must exist an element y in G of order 4 with $y^2 = x$. The irreducible real representations of $\langle y \rangle$ have degree 1 or 2, and the representations $\chi\colon \langle y \rangle \to O(1)$ satisfy $\chi(y^2) = 1$. As $\rho(y^2) = -I$, $\rho | \langle y \rangle$ cannot contain 1-dimensional representations. Hence $\deg \rho = m$ is even. If p is an odd prime and $H \leq G$ has order p, then $\rho | H$ must be free since H has odd order and ρ has even degree. Thus we have shown that the restriction of ρ to any subgroup of prime order is free. This implies that ρ is free. □

REMARK. The classification of groups G having a free representation can be summarized as follows: The maximal normal subgroup K of G of odd order is metacyclic and the quotient G/K belongs to one of the types

$$\text{I}\quad Z_{2^k} \qquad \text{II}\quad Q_{2^k} \qquad \text{III}\quad SL_2(3)$$
$$\text{IV}\quad TL_2(3) \qquad \text{V}\quad SL_2(5) \qquad \text{VI}\quad TL_2(5),$$

where Q_{2^k} is the generalized quaternion group and $TL_2(p)$ contains $SL_2(p)$ as a subgroup of index 2 [**14**, Theorems 6.1.11, 6.3.1]. A representation of the form $\psi + 1$, where ψ is free, is called semifree. It follows that G admits a nontrivial homomorphism $\chi\colon G \to Z_2$ and hence pseudofree, not semifree representations precisely when G belongs to types I (and has even order), II, IV or VI.

3. Representations at fixed points. We shall prove

THEOREM 1. *Let G be a finite group acting smoothly on a homotopy sphere Σ^m, $m \geq 5$, such that there is a fixed point $a \in \Sigma^G$ at which the local representation ρ_a of G is 2-pseudofree. Then*

(i) *the representations of G at any two fixed points are topologically equivalent,*

(ii) *if there are at least two fixed points, then Σ^m is G-homeomorphic to the linear representation sphere $S^m(\rho_a \oplus 1)$,*

(iii) *if there is only one fixed point, then $G = A_5$ and $\Sigma^m = S^6$.*

REMARK. Petrie announced in [**10**] that $A_5 = PSL_2(5)$ can act smoothly on a homotopy sphere with precisely one fixed point. However, the dimensions of his examples are far higher than 6 [**10**, 4.20, p. 185]. We do not know whether case (iii) can occur. E. Stein produces in [**13**] a smooth action of the binary icosahedral group $SL_2(5)$ on S^7 with only one fixed point. This action is 3-pseudofree.

PROOF. We prove first statement (ii). Let us assume that G has at least two fixed points $a, b \in \Sigma^G$. By Illman's Theorem 5 [**6**, p. 144] (ii) follows if we can show that $\Sigma^H \cong S^k$, $k = 0, 1$ or 2 for each subgroup $H \neq \{e\}$.

Assume inductively the claim proved for all groups H having order less than the order of G. If there is a nontrivial normal subgroup $H \lhd G$, $\{e\} \neq H \neq G$, then by induction $\Sigma^H \cong S^{k'}$, $k' = 0, 1$ or 2, and the action of G/H must be linear, so $\Sigma^G \cong S^k$, $k = 0, 1$ or 2. Hence it is enough to prove that $\Sigma^G \cong S^k$, $k = 0, 1$ or 2 for all simple groups G, for which $\Sigma^H \cong S^k$, $k = 0, 1$ or 2 for subgroups H, $\{e\} \neq H \neq G$.

If $G = Z_p$, the claim follows by Smith theory even without the assumption of two fixed points (see [**6**, Proposition 1, p. 140]). If G is a nonabelian simple group, it contains a dihedral group $H \leq G$. Since the restriction of ρ_a to H is 2-pseudofree and $m \geq 5$ by assumption, it follows from Lemma 2.1 that $m = 6$ and $(T_a\Sigma)^H = \{0\}$. Hence Σ^H is 0-dimensional and, by the induction assumption, $\Sigma^H \cong S^0$ so we must have $\Sigma^H = \{a, b\}$. Then

$$\{a, b\} \subset \Sigma^G \subset \Sigma^H = \{a, b\}$$

so that $\Sigma^G = \{a, b\} \cong S^0$. This proves statement (ii).

If there is only one fixed point, statement (i) about comparing the representations at different fixed points is trivially valid. In the case of at least two fixed points (i) follows from (ii) as in [**6**, Corollary B, p. 154].

We are left with (iii). Assume that G has only one fixed point a. We have to prove that $G = A_5$ and $\Sigma^m = S^6$. If a normal subgroup $H \neq \{e\}$ has at least two fixed points, then it follows from (ii) that $\Sigma^H \cong S^{k'}$, $k' = 0, 1$ or 2, and then $\Sigma^G \cong S^k$, $k = 0, 1$ or 2, as above. This contradicts the assumption $\Sigma^G = \{a\}$. Hence $\Sigma^H = \{a\}$ for each normal $H \lhd G$. Choose a minimal normal subgroup K of G. Then K is a direct product of isomorphic simple groups [**2**, Theorem 2.1.5, p. 17]. Choose a simple factor H of K. Then $K \lhd G$ implies $\Sigma^K = \{a\}$ and $H \lhd K$ implies $\Sigma^H = \{a\}$. By Smith theory H cannot be cyclic of prime order since $\chi(\Sigma^{Z_p}) = \chi(\Sigma) \pmod p$ so H must be a simple nonabelian group. Especially, it follows that if G is solvable, then it has at least two fixed points.

Since H has a 2-pseudofree representation of dimension ≥ 5, $m = 6$ by Corollary 2.2. Then Σ^6 is the standard 6-sphere S^6 since S^6 admits only one differentiable structure. It now suffices to show the following proposition.

PROPOSITION 3.1. (i) *If a simple group G acts smoothly on S^6 with only one fixed point a such that the representation at a is 2-pseudofree, then $G = A_5$.*

(ii) *If A_5 acts on S^6 as above and A_5 is a proper normal subgroup of G, then the action of A_5 does not extend to an action of G with a 2-pseudofree representation at a.*

PROOF. We first settle (ii). It is enough to consider the case when $A_5 \triangleleft G$ and $(G : A_5)$ is a prime p. Since $\mathrm{Aut}(A_5) = S_5$, we have in fact $G = S_5$ or $G = A_5 \times Z_p$. It can be checked from the character table of S_5 that S_5 admits no 6-dimensional 2-pseudofree representations. On the other hand if $G = A_5 \times Z_p$, then we have $(S^6)^{Z_p} \cong S^2, S^1$ or S^0.

The smooth action of $G/Z_p = A_5$ on $(S^6)^{Z_p}$ must be trivial since it has a fixed point a and A_5 has only trivial representations in degrees ≤ 2. Then $(S^6)^G = (S^6)^{Z_p}$ contains at least two fixed points, hence $(S^6)^{A_5} \neq \{a\}$ contradicting the assumptions. This proves (ii).

If a simple group G acts on S^6 as in (i), then G has a 2-pseudofree representation in degree 6. By Corollary 2.2, only $G = A_5$ has 1-pseudofree representations and these have degree 3. Hence either $G = A_5$ or the representation $\rho_a \colon G \to SO(6)$ is irreducible as a real representation. Then ρ_a is either irreducible as a complex 6-dimensional representation or it is the realification of a complex irreducible 3-dimensional representation which is 1-pseudofree in the complex sense.

Now we may apply the classification of irreducible primitive unimodular linear groups of degrees 3 and 6 given in Blichfeldt [**1**] and Lindsey [**8**]. It is clear that every complex representation of a nonabelian simple group is unimodular, that is $\rho(G) \subset SU(m)$. If G is simple and admits an imprimitive faithful irreducible complex G-module of degree ≤ 6, then G is a simple primitive permutation group of degree ≤ 6, hence $G \leq A_6$ and in fact either $G = A_5$ or A_6 [**5**, II, 4.7].

Thus it is enough to consider A_5, A_6 and the simple irreducible primitive unimodular groups of degree 3 or 6. The results are as follows. The simple groups

$$A_7, \quad PSL_2(7), \quad PSU_4(2), \quad U_3(3)$$

have an irreducible complex representation in degree 6. From the character tables [**7**] one can check that the representations of the first three groups are not 2-pseudofree. $U_3(3)$ has a 2-pseudofree representation of degree 6 in the complex sense, but this representation is quaternionic, i.e. it corresponds to a 12-dimensional real representation.

The simple groups $PSL_2(7)$ and A_5 have irreducible complex representations in degree 3. $PSL_2(7)$ has two complex conjugate irreducible representations in degree 3 and their sum is the unique 2-pseudofree representation $\rho \colon PSL_2(7) \to SO(6)$. Let $G = PSL_2(7)$ and let V be the corresponding 6-dimensional real G-module.

We claim that if $G = PSL_2(7)$ acts smoothly on $\Sigma = S^6$ with a fixed point at which the representation is V, then G has two fixed points. Indeed, G is isomorphic to $PSL_3(2)$ [**5**, II, 6.14] and acts transitively on the seven points of the projective plane over Z_2. As $|G| = 168$, the isotropy group of any point has order 24 (they are actually isomorphic to S_4). Let H_1 and H_2 be the isotropy groups of two different points on the line at infinity. Since $H_1 \neq H_2$ and they have prime index in G, $G = \langle H_1, H_2 \rangle$. Then $H = H_1 \cap H_2$ fixes also the third point on the line at infinity and consists therefore of the translations of the affine plane, i.e. $H \cong Z_2 \oplus Z_2$. By Lemma

2.1, $V^H = \{0\}$, hence also $V^{H_1} = V^{H_2} = \{0\}$. Since H, H_1 and H_2 are solvable, each space Σ^H, Σ^{H_1} and Σ^{H_2} consists of two points. As $\Sigma^{H_i} \subset \Sigma^H$, $i = 1, 2$, we must have $\Sigma^H = \Sigma^{H_1} = \Sigma^{H_2} \cong S^0$. But H_1 and H_2 generate G. We conclude that $\Sigma^G = \Sigma^{H_1} \cap \Sigma^{H_2} \cong S^0$.

This completes the proof of Proposition 3.1 and Theorem 1. □

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