THE NORMAL EXTENSIONS OF SUBGROUP TOPOLOGIES

BRADD CLARK AND VICTOR SCHNEIDER

Abstract. Let $H$ be a topological group contained in a group $G$. A topology which makes $G$ a topological group inducing the given topology on $H$ is called an extending topology. The set of all extending topologies forms a complete semilattice in the lattice of group topologies on $G$. The structure of this semilattice is studied by considering normal subgroups which intersect $H$ in the identity.

Let $G$ be a group and $\mathcal{L}$ the collection of continuous topologies on $G$. If $\{T_\alpha\}_{\alpha \in \Gamma}$ is any collection of topologies in $\mathcal{L}$, we can create a new topological group $\mathcal{G} = \prod_{\alpha \in \Gamma} (G, T_\alpha)$. The embedding of $G$ into $\mathcal{G}$ along the diagonal is an algebraic embedding. The relative topology on $G$ in $\mathcal{G}$ is the supremum topology on $G$ relative to $\{T_\alpha\}_{\alpha \in \Gamma}$. Since $G$ is a topological group we see that the subgroup $G$ is also a topological group when given this topology. Hence $\mathcal{L}$ is closed under the operation $T_\alpha \vee T_\beta$ where $T_\alpha \vee T_\beta$ denotes the supremum topology $T_\alpha$ and $T_\beta$.

Now suppose $T_\alpha$ and $T_\beta$ are in $\mathcal{L}$ and $\mathcal{B} \subseteq \mathcal{L}$ is the collection of all topologies $T_\gamma$ in $\mathcal{L}$ that satisfy $T_\gamma \subseteq T_\alpha$ and $T_\gamma \subseteq T_\beta$. $\mathcal{B} \neq \emptyset$ since the indiscrete topology on $G$ is in $\mathcal{L}$. We define $T_\alpha \wedge T_\beta = (\bigvee_{\gamma \in \mathcal{B}} T_\gamma)$. Certainly $\mathcal{L}$ is closed under $\wedge$ and hence $(\mathcal{L}, \vee, \wedge)$ forms a complete lattice. It should be noted that this lattice is different from the usual lattice of topologies on a set $X$ since the intersection of two group topologies may not be a group topology.

Let $H$ be a topological group contained in a group $G$. A topology which makes $G$ a topological group inducing the given topology on $H$ is called an extending topology. The set of all extending topologies from $H$ to $G$, $\mathcal{E}$, is a complete subsemilattice of the complete lattice of group topologies on $G$. The purpose of this paper is to further study $\mathcal{E}$.

Definition. A group topology for $H$ is said to be translatable if and only if for every neighborhood $U$ of $e$ in $H$ and every $g \in G$, the set $gUg^{-1}$ contains a neighborhood of $e$ in $H$. If $t$ is a translatable topology on $H$ we define the translation topology $T_t^H$ to be that topology on $G$ which has $\{gU | g \in U \text{ and } U \in t\}$ as a basis.

One way to create a topology on $G$ is to find a homomorphism from $G$ to a topological group $G'$. The weak topology on $G$ relative to this homomorphism will make $G$ into a topological group also. Occasionally this topology will be an extending topology. Let $H$ be a subgroup of $G$ endowed with the translatable
topology $t$. Let $f$ be a homomorphism from $G$ onto the topological group $G'$ and let $H' = f(H)$. Let $E$ be the closure of $\{e\}$ in $H$.

**THEOREM 1.** The weak topology on $G$ induced by $f$ is an extending topology for $H$ if and only if $f$ is continuous and open (as a map from $H$ to $H'$) and $H \cap \ker f \subseteq E$.

**Proof.** Let $T$ be the weak topology on $G$ induced by $f$ and suppose that $T \in \mathcal{E}$. If $x \in H \cap \ker f$ and $U$ is an open neighborhood of $x$ in $H$, then there exists an open set $U'$ in $H'$ such that $U = H \cap f^{-1}(U')$. But $e' \in U'$ and hence $e \in U$.

Now suppose that $H \cap \ker f \subseteq E$ and $f$ is continuous and open as a map from $H$ to $H'$. Let $U \in t$. Then $U' = f(U)$ is open in $H'$. Let $x \in H \cap f^{-1}(U')$. We can find a $y \in U$ such that $f(x) = f(y)$ and hence $xy^{-1} \in \ker f$. So if $xy^{-1} \in V$ and $V \in t$, then $e \in V$. But this means that $x \in U$ and hence $U = H \cap f^{-1}(U')$.

Let $N = \ker f$. Obviously, $N$ is a normal subgroup with $N \cap H = \{e\}$. Such subgroup structure occurs frequently (e.g., if $G$ has a presentation of the form $\{a, b|a^n = b^m\}$, then $H = [G, G]$ and $N = Z(G)$ satisfy this condition). Since we are trying to create new extending topologies on $G$, we may hope that any extending topology on $G$ will induce a topology on $G/N$ that can be used to create a weak topology on $G$ that also extends the topology on $H$. In general this will not work since we cannot be sure that $f|_H$ is a homeomorphism where $f: G \to G/N$ is the natural map.

As an example of this, suppose that $G = \mathbb{R}^1$ and $N = \mathbb{Q}$. If $H$ is a linear complement to $\mathbb{Q}$ when $\mathbb{R}^1$ is regarded as a vector space over $\mathbb{Q}$, then the usual topology on $H$ will be translatable since $\mathbb{R}^1$ is abelian. But the weak topology on $\mathbb{R}^1$ relative to the natural map $f: \mathbb{R}^1 \to \mathbb{R}^1/\mathbb{Q}$ is the indiscrete topology.

Suppose that $H$ is any subgroup and that $N$ is a normal subgroup with the property that $H \cap N = \{e\}$. Certainly the multiplication map $H \times N \to HN$ is bijective. If $H$ and $N$ are assigned group topologies, then this map induces a topology on $HN$, called the product topology, which may or may not be a group topology for $HN$. We note that a group topology for $HN$ is a product topology if and only if the map of $H$ to $HN/N$ is a homeomorphism.

**THEOREM 2.** Suppose $H$ and $N$ are endowed with topologies such that the product topology for $HN$ is a group topology. Then this topology for $HN$ is translatable if the given topologies for both $H$ and $N$ are translatable. On the other hand, if the topology for $HN$ is translatable, then the topology for $N$ is translatable.

We observe that the indiscrete topology on a normal subgroup is always translatable. If $H$ is normal, then the product topology for $HN$ will be a group topology. So if $t$ is a translatable topology for $H$ and $N \cap H = \{e\}$, then there is an extending topology which has the indiscrete topology as the relative topology on $N$.

The collection of normal subgroups $\mathcal{N}$ with the property that if $n \in \mathcal{N}$ then $n \cap H = \{e\}$ is a nonempty collection. Thus if we partially order $\mathcal{N}$ by set inclusion, we can use Zorn's Lemma to find maximal normal subgroups $N$ with $N \cap H = \{e\}$. As we shall soon see, these maximal subgroups will give us the best insight to the structure of $\mathcal{E}$. 
Let $H$ be normal. Then $T \in \mathcal{E}$ where $T$ is generated using $\{ gU | U \in \mathcal{T} \times \{ N, \emptyset \}, g \in G \}$ as a basis. Let $\tau_N$ be the topology on $G/H$ which is generated by using the natural map from $(G, T)$ to $(G/H, \tau_N)$. The set of topologies on $G/H$ which make $G/H$ into a topological group and which lie between $\tau_N$ and the discrete topology on $G/H$ forms a lattice $L_N$ contained in the lattice of all continuous topologies that exist on $G/H$. If $n$ is a normal subgroup of $G$ with $n \subset N$, then a moment’s reflection will show that $L_n \subset L_N$ where $L_n$ is defined in the same fashion as $L_N$.

**Theorem 3.** If $N \subset G$ is a normal subgroup with $N \cap H = \{ e \}$, then the semi-lattice $\mathcal{E}$ contains a sublattice isomorphic to $L_N$.

**Proof.** As we have seen, $T \in \mathcal{E}$ and the natural map from $(G, T)$ to $(G/H, \tau_N)$ puts a continuous topology on $G/H$. By [1] we know that for every topology $\tau$ in $L_N$ there is a unique topology in $\mathcal{E}$ that induces $\tau$ and is finer than $T$. Let $\mathcal{L}_N$ be this collection of topologies. Since $\mathcal{L}_N$ includes both $T$ and the translation topology and since we have an order-preserving bijection between $\mathcal{L}_N$ and $L_N$, it is clear that $\mathcal{L}_N$ is isomorphic to $L_N$.

As an example of this theorem’s applications, let $G$ be the torus knot group with the presentation $\{ a, b | a^n = b^m \}$ with $(n, m) = 1$. We place the residually finite topology on $G$ as described in [2]. If $H = [G, G]$, we then have a relative topology defined on $H$ which we call $\iota$. Let $N$ be the center of $G$. Of course, any group topology placed on the center of a group will be a translatable topology. In this example $N$ is isomorphic to $\mathbb{Z}$ and clearly $N \cap H = \{ e \}$. $G/H \cong \mathbb{Z}$ is the homology group associated with the torus knot space, and the image of $N$ in $G/H$ is the subgroup $\{ nmx | x \in \mathbb{Z} \}$. The indiscrete topology on $N$ and hence the indiscrete topology on the subgroup $\{ nmx | x \in \mathbb{Z} \}$ translates to make $G$ and $G/H$ into topological groups. Let $\{ p_1, p_2, \ldots, p_j \}$ be the collection of primes that divide the composite number $nm$, and let $\mathcal{P}$ be any collection of primes that contain $\{ p_1, p_2, \ldots, p_j \}$ as a subset. There is a unique topology $\tau_{\mathcal{P}}$ on $\mathbb{Z}$ which is the supremum topology of $\{ p$-adic topology $| p \in \mathcal{P} \}$. If $\mathcal{P} \neq \mathcal{P}'$, then $\tau_{\mathcal{P}} \neq \tau_{\mathcal{P}'}$. Therefore $\mathcal{E}$ contains uncountably many topologies corresponding to the $\tau_{\mathcal{P}}$ topologies on $\mathbb{Z}$. Of course, there are coarser topologies on $G/H$ which fail to be Hausdorff, but which also have their corresponding topology in $\mathcal{E}$. As pointed out in [3], there are topologies on $G/H$ which are finer than the subgroup topology, but which are not the discrete topology. They also have their counterparts in $\mathcal{E}$. Although we have discovered many topologies in $\mathcal{L}_N \subset \mathcal{E}$ and their relationship to each other, we have not found the ultimate structure of $\mathcal{E}$. In this example we note that the residually finite topology on $G$ is not an element of $\mathcal{L}_N$.

If $T \in \mathcal{L}_N \subset \mathcal{E}$ for some normal subgroup $N \subset G$, we shall call $T$ a normal extension topology. One might hope that every topology in $\mathcal{E}$ is a normal extension topology. This is unfortunately not the case. Let $G = \mathbb{R}^1$ and $H = \mathbb{Z} \subset \mathbb{R}^1$. We place the discrete topology $\iota$ on $\mathbb{Z}$ and consider the semilattice $\mathcal{E}$.

Let $N$ be a linear complement to $\mathbb{Q}$ when $\mathbb{R}^1$ is regarded as a vector space over $\mathbb{Q}$. Suppose that $n/m \in N \cap \mathbb{Q}$. Certainly $n \in N \cap \mathbb{Z}$ and hence $N \cap \mathbb{Q} = \{ 0 \}$. The coarsest normal extension topology we can obtain on $\mathbb{R}^1$ using $N$ is the topology
obtained by placing the indiscrete topology on $N$, the discrete topology on $Z$, the resulting product topology on $N \times Z$ and translating this topology throughout $R^1$. But if we place the indiscrete topology on $N$, the usual topology on $Q$, the resulting product topology on $N \times Q$ and translate this topology throughout $R^1$, we will obtain a coarser topology.

The authors wish to thank Douglass Grant and the referee for helpful comments with respect to this paper.

REFERENCES