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THE NORMAL EXTENSIONS OF SUBGROUP TOPOLOGIES

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ABSTRACT. Let H be a topological group contained in a group G. A topology which makes G a topological group inducing the given topology on H is called an extending topology. The set of all extending topologies forms a complete semilattice in the lattice of group topologies on G. The structure of this semilattice is studied by considering normal subgroups which intersect H in the identity.

Let G be a group and $\mathscr{L}$ the collection of continuous topologies on G. If $\{T_\alpha\}_{\alpha \in \Gamma}$ is any collection of topologies in $\mathscr{L}$, we can create a new topological group $\mathscr{G} = \prod_{\alpha \in \Gamma}(G, T_\alpha)$. The embedding of G into $\mathscr{G}$ along the diagonal is an algebraic embedding. The relative topology on G in $\mathscr{G}$ is the supremum topology on G relative to $\{T_\alpha\}_{\alpha \in \Gamma}$. Since G is a topological group we see that the subgroup G is also a topological group when given this topology. Hence $\mathscr{L}$ is closed under the operation $T_\alpha \vee T_\beta$ where $T_\alpha \vee T_\beta$ denotes the supremum topology T_α and T_β.

Now suppose T_α and T_β are in $\mathscr{L}$ and $\mathscr{B} \subset \mathscr{L}$ is the collection of all topologies T_γ in $\mathscr{L}$ that satisfy $T_\gamma \subseteq T_\alpha$ and $T_\gamma \subseteq T_\beta$. $\mathscr{B} \neq \varnothing$ since the indiscrete topology on G is in $\mathscr{L}$. We define $T_\alpha \wedge T_\beta = (\bigvee_{\gamma \in \mathscr{B}} T_\gamma)$. Certainly $\mathscr{L}$ is closed under $\wedge$ and hence $(\mathscr{L}, \vee, \wedge)$ forms a complete lattice. It should be noted that this lattice is different from the usual lattice of topologies on a set X since the intersection of two group topologies may not be a group topology.

Let H be a topological group contained in a group G. A topology which makes G a topological group inducing the given topology on H is called an extending topology. The set of all extending topologies from H to G, $\mathscr{E}$, is a complete subsemilattice of the complete lattice of group topologies on G. The purpose of this paper is to further study $\mathscr{E}$.

DEFINITION. A group topology for H is said to be *translatable* if and only if for every neighborhood U of e in H and every $g \in G$, the set gUg^{-1} contains a neighborhood of e in H. If t is a translatable topology on H we define the *translation topology* T_H^* to be that topology on G which has $\{gU \mid g \in U \text{ and } U \in t\}$ as a basis.

One way to create a topology on G is to find a homomorphism from G to a topological group G'. The weak topology on G relative to this homomorphism will make G into a topological group also. Occasionally this topology will be an extending topology. Let H be a subgroup of G endowed with the translatable

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topology t. Let f be a homomorphism from G onto the topological group G' and let $H' = f(H)$. Let E be the closure of $\{e\}$ in H.

THEOREM 1. *The weak topology on G induced by f is an extending topology for H if and only if f is continuous and open (as a map from H to H') and* $H \cap \ker f \subseteq E$.

PROOF. Let T be the weak topology on G induced by f and suppose that $T \in \mathscr{E}$. If $x \in H \cap \ker f$ and U is an open neighborhood of x in H, then there exists an open set U' in H' such that $U = H \cap f^{-1}(U')$. But $e' \in U'$ and hence $e \in U$.

Now suppose that $H \cap \ker f \subseteq E$ and f is continuous and open as a map from H to H'. Let $U \in t$. Then $U' = f(U)$ is open in H'. Let $x \in H \cap f^{-1}(U')$. We can find a $y \in U$ such that $f(x) = f(y)$ and hence $xy^{-1} \in \ker f$. So if $xy^{-1} \in V$ and $V \in t$, then $e \in V$. But this means that $x \in U$ and hence $U = H \cap f^{-1}(U')$.

Let $N = \ker f$. Obviously, N is a normal subgroup with $N \cap H = \{e\}$. Such subgroup structure occurs frequently (e.g., if G has a presentation of the form $\{a, b | a^n = b^m\}$, then $H = [G, G]$ and $N = Z(G)$ satisfy this condition). Since we are trying to create new extending topologies on G, we may hope that any extending topology on G will induce a topology on G/N that can be used to create a weak topology on G that also extends the topology on H. In general this will not work since we cannot be sure that $f|_H$ is a homeomorphism where $f\colon G \to G/N$ is the natural map.

As an example of this, suppose that $G = \mathbf{R}^1$ and $N = \mathbf{Q}$. If H is a linear complement to $\mathbf{Q}$ when $\mathbf{R}^1$ is regarded as a vector space over $\mathbf{Q}$, then the usual topology on H will be translatable since $\mathbf{R}^1$ is abelian. But the weak topology on $\mathbf{R}^1$ relative to the natural map $f\colon \mathbf{R}^1 \to \mathbf{R}^1/Q$ is the indiscrete topology.

Suppose that H is any subgroup and that N is a normal subgroup with the property that $H \cap N = \{e\}$. Certainly the multiplication map $H \times N \to HN$ is bijective. If H and N are assigned group topologies, then this map induces a topology on HN, called the product topology, which may or may not be a group topology for HN. We note that a group topology for HN is a product topology if and only if the map of H to HN/N is a homeomorphism.

THEOREM 2. *Suppose H and N are endowed with topologies such that the product topology for HN is a group topology. Then this topology for HN is translatable if the given topologies for both H and N are translatable. On the other hand, if the topology for HN is translatable, then the topology for N is translatable.*

We observe that the indiscrete topology on a normal subgroup is always translatable. If H is normal, then the product topology for HN will be a group topology. So if t is a translatable topology for H and $N \cap H = \{e\}$, then there is an extending topology which has the indiscrete topology as the relative topology on N.

The collection of normal subgroups $\mathscr{N}$ with the property that if $n \in \mathscr{N}$ then $n \cap H = \{e\}$ is a nonempty collection. Thus if we partially order $\mathscr{N}$ by set inclusion, we can use Zorn's Lemma to find maximal normal subgroups N with $N \cap H = \{e\}$. As we shall soon see, these maximal subgroups will give us the best insight to the structure of $\mathscr{E}$.

Let H be normal. Then $T \in \mathscr{E}$ where T is generated using $\{gU | U \in t \times \{N, \varnothing\}, g \in G\}$ as a basis. Let τ_N be the topology on G/H which is generated by using the natural map from (G, T) to $(G/H, \tau_N)$. The set of topologies on G/H which make G/H into a topological group and which lie between τ_N and the discrete topology on G/H forms a lattice L_N contained in the lattice of all continuous topologies that exist on G/H. If n is a normal subgroup of G with $n \subset N$, then a moment's reflection will show that $L_n \subset L_N$ where L_n is defined in the same fashion as L_N.

THEOREM 3. *If $N \subset G$ is a normal subgroup with $N \cap H = \{e\}$, then the semilattice $\mathscr{E}$ contains a sublattice isomorphic to L_N.*

PROOF. As we have seen, $T \in \mathscr{E}$ and the natural map from (G, T) to $(G/H, \tau_N)$ puts a continuous topology on G/H. By [**1**] we know that for every topology τ in L_N there is a unique topology in $\mathscr{E}$ that induces τ and is finer than T. Let $\mathscr{L}_N$ be this collection of topologies. Since $\mathscr{L}_N$ includes both T and the translation topology and since we have an order-preserving bijection between $\mathscr{L}_N$ and L_N, it is clear that $\mathscr{L}_N$ is isomorphic to L_N.

As an example of this theorem's applications, let G be the torus knot group with the presentation $\{a, b | a^n = b^m\}$ with $(n, m) = 1$. We place the residually finite topology on G as described in [**2**]. If $H = [G, G]$, we then have a relative topology defined on H which we call t. Let N be the center of G. Of course, any group topology placed on the center of a group will be a translatable topology. In this example N is isomorphic to $\mathbf{Z}$ and clearly $N \cap H = \{e\}$. $G/H \cong \mathbf{Z}$ is the homology group associated with the torus knot space, and the image of N in G/H is the subgroup $\{nmx | x \in \mathbf{Z}\}$. The indiscrete topology on N and hence the indiscrete topology on the subgroup $\{nmx | x \in \mathbf{Z}\}$ translates to make G and G/H into topological groups. Let $\{p_1, p_2, \ldots, p_j\}$ be the collection of primes that divide the composite number nm, and let $\mathscr{P}$ be any collection of primes that contain $\{p_1, p_2, \ldots, p_j\}$ as a subset. There is a unique topology $\tau_{\mathscr{P}}$ on $\mathbf{Z}$ which is the supremum topology of $\{p\text{-adic topology} \mid p \in \mathscr{P}\}$. If $\mathscr{P} \neq \mathscr{P}'$, then $\tau_{\mathscr{P}} \neq \tau_{\mathscr{P}'}$. Therefore $\mathscr{E}$ contains uncountably many topologies corresponding to the $\tau_{\mathscr{P}}$ topologies on $\mathbf{Z}$. Of course, there are coarser topologies on G/H which fail to be Hausdorff, but which also have their corresponding topology in $\mathscr{E}$. As pointed out in [**3**], there are topologies on G/H which are finer than the subgroup topology, but which are not the discrete topology. They also have their counterparts in $\mathscr{E}$. Although we have discovered many topologies in $\mathscr{L}_N \subset \mathscr{E}$ and their relationship to each other, we have not found the ultimate structure of $\mathscr{E}$. In this example we note that the residually finite topology on G is not an element of $\mathscr{L}_N$.

If $T \in \mathscr{L}_N \subset \mathscr{E}$ for some normal subgroup $N \subset G$, we shall call T a *normal extension topology*. One might hope that every topology in $\mathscr{E}$ is a normal extension topology. This is unfortunately not the case. Let $G = \mathbf{R}^1$ and $H = \mathbf{Z} \subset \mathbf{R}^1$. We place the discrete topology t on $\mathbf{Z}$ and consider the semilattice $\mathscr{E}$.

Let N be a linear complement to $\mathbf{Q}$ when $\mathbf{R}^1$ is regarded as a vector space over $\mathbf{Q}$. Suppose that $n/m \in N \cap \mathbf{Q}$. Certainly $n \in N \cap \mathbf{Z}$ and hence $N \cap \mathbf{Q} = \{0\}$. The coarsest normal extension topology we can obtain on $\mathbf{R}^1$ using N is the topology obtained by placing the indiscrete topology on N, the discrete topology on $\mathbf{Z}$, the resulting product topology on $N \times \mathbf{Z}$ and translating this topology throughout $\mathbf{R}^1$. But if we place the indiscrete topology on N, the usual topology on $\mathbf{Q}$, the resulting product topology on $N \times \mathbf{Q}$ and translate this topology throughout $\mathbf{R}^1$, we will obtain a coarser topology.

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