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SHORTER NOTES

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A REMARK ON THE NONUNIQUENESS OF TANGENT CONES

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ABSTRACT. We give an example of an n-dimensional varifold V such that the n-dimensional density of V at some point p equals 1, the $(n-1)$ density of the first variation at p equals 0, and V has nonunique tangent cones at p. This answers the question implicit in [W. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491, §6.5].

The following simple example answers the question implicit in [W. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491, §6.5].

Let V be the varifold in $\mathbf{R}^{n+1}$ corresponding to the graph of the function $f\colon B^n_{1/10} \to \mathbf{R}$, where $B^n_{1/10} = \{x \in \mathbf{R}^n\colon |x| \le 1/10\}$ and $n \ge 2$, given by

$$f(x) = x_1 \sin\log\log|x|^{-1} \text{ if } x \neq 0, \quad f(0) = 0.$$

Then
(a) $\Theta^n(\|V\|, 0) = 1$,
(b) $\Theta^{n-1}(\|\delta V\|, 0) = 0$,
(c) T is a tangent cone to V at 0 iff T is the varifold corresponding to the graph of the function $\{x \mapsto \alpha x_1\}$ for some $|\alpha| \le 1$.

To see this let V_ρ be the varifold corresponding to the graph of f_ρ where $f_\rho(x) = \rho^{-1} f(\rho x)$, $\rho > 0$. Thus V_ρ is just the "blow-up" of V at 0 by the factor ρ^{-1}.

One checks that

$$(1)\qquad D_i f(x) = \delta_{i1} \sin\log\log(\rho|x|)^{-1} + x_1 x_i |x|^{-2} \cos\log\log(\rho|x|)^{-1} (\log\rho|x|)^{-1},$$

$$(2)\qquad |D_{ij} f_\rho(x)| \le c|x|^{-1} |\log\rho|x||^{-1},$$

and so

$$(3)\qquad f_\rho \in W^{2,n},$$

while clearly

$$(4)\qquad f_\rho \text{ are uniformly Lipschitz},$$

all for $|x| \le 1/10$, $0 < \rho \le 1$, say.

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Next let $\rho_j = (\exp\exp T_j)^{-1} \to 0$ and pass to a subsequence so that $\sin T_j \to \alpha$, say. For each $\varepsilon > 0$ it is easy to check that

$$\sin\log\log(\rho_j|x|)^{-1} \to \alpha \tag{5}$$

uniformly for $\varepsilon \le |x| \le 1$.

{To see this just note that for any $\delta > 0$

$$[\exp\exp(T_j + \delta)]^{-1} \le \varepsilon(\exp\exp T_j)^{-1} = \varepsilon\rho_j$$

for $j \ge J(\delta, \varepsilon)$, say. Hence for $\varepsilon \le |x| \le 1$

$$T_j = \log\log\rho_j^{-1} \le \log\log(\rho_j|x|)^{-1} \le \log\log(\varepsilon\rho_j)^{-1} \le T_j + \delta.$$

Hence

$$|\sin\log\log(\rho_j|x|)^{-1} - \sin T_j| \le \delta$$

for $\varepsilon \le |x| \le 1$, $j \ge J(\delta, \varepsilon)$. The result follows.}

From (5), the definition of f_ρ, and (1), it follows that

$$f_\rho(x) \to \alpha x_1 \tag{6}$$

uniformly in the C^1 norm for $\varepsilon \le |x| \le 1$.

Then (a) and (c) each follow from (4) and (6). To see (b) just note that if H is the mean curvature vector of the graph of f and $B_\rho = \{z \in \mathbf{R}^{n+1}\colon |z| \le 1\}$, then

$$\rho^{1-n}\|\delta V\|(B_\rho) = \rho^{1-n}\int_{V\cap B_\rho} |H|\,d\|V\|$$

(using (4) and Stokes' theorem to take care of the singularity at 0, noting in particular that $\mathcal{H}^{n-1}\{(x, f(x))\colon |x| = \delta\} \to 0$ as $\delta \to 0$),

$$\le c\left(\int_{V\cap B_\rho} |H|^n\,d\|V\|\right)^{1/n}$$

(using (a))

$$\to 0 \quad \text{as } \rho \to 0$$

(using (3) and (4)).

Finally let us remark that if $f \in W^{2,p}_{\text{loc}}$ for some $p > n$, then $f \in C^{1,\alpha}_{\text{loc}}$ and so tangent cones exist, are planes, and are unique.

If $n = 1$ and $f \in W^{2,1}_{\text{loc}}$ then Df is absolutely continuous and so tangent cones again exist, are planes, and are unique.

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