THE nTH ROOTS OF SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

WILLIAM A. HARRIS, JR. AND YASUTAKA SIBUYA

ABSTRACT. In this paper we shall prove the following theorem: Let $K$ be a differential field of characteristic zero. Let $\varphi$ and $\psi$ be elements of a differential field extension of $K$ such that (i) $\varphi \neq 0$ and $\psi \neq 0$; (ii) $\varphi$ and $\psi$ satisfy nontrivial linear differential equations with coefficients in $K$, say, $P(\varphi) = 0$ and $Q(\psi) = 0$; (iii) $\varphi = \psi^n$ for some positive integer $n$ such that $n \geq \text{ord } P$. Then the logarithmic derivatives of $\varphi$ and $\psi$ are algebraic over $K$. (Note that $\varphi'/\varphi = n(\psi'/\psi).$)

1. Introduction. Let $K$ be a differential field of characteristic zero. We denote by $\mathcal{D}_K$ the ring of linear ordinary differential operators with coefficients in $K$, that is

$$\mathcal{D}_K = \left\{ \sum_{k=0}^{m} a_k D^k; a_k \in K, m \in N \right\},$$

where $D$ denotes differentiation in any differential field extension of $K$ and $N$ is the set of all nonnegative integers. In a previous paper [2] we characterized those functions which together with their reciprocal satisfy linear differential equations:

**THEOREM A.** Let $\varphi$ be an element of a differential field extension of $K$ such that

(i) $\varphi \neq 0$;
(ii) $P(\varphi) = 0$ for some $P \in \mathcal{D}_K - \{0\}$;
(iii) $Q(1/\varphi) = 0$ for some $Q \in \mathcal{D}_K - \{0\}$.

Then, the logarithmic derivative of $\varphi$ ($= \varphi'/\varphi$) is algebraic over $K$.

In the present paper, utilizing a similar method, we shall prove the following result:

**THEOREM B.** Let $\varphi$ and $\psi$ be elements of a differential field extension of $K$ such that

(i) $\varphi \neq 0$ and $\psi \neq 0$;
(ii) $P(\varphi) = 0$ for some $P \in \mathcal{D}_K - \{0\}$;
(iii) $Q(\psi) = 0$ for some $Q \in \mathcal{D}_K - \{0\}$;
(iv) $\varphi = \psi^n$ for some positive integer $n$ such that

$$(1.1) \quad n \geq \text{ord } P.$$

Then $\varphi'/\varphi = n(\psi'/\psi)$ is algebraic over $K$. 

Received by the editors June 24, 1985.

1980 Mathematics Subject Classification. Primary 12H05, 13B20, 13J05, 34A30.

1This paper was presented in a special session on ordinary differential equations at the 91st annual meeting of the AMS on January 10, 1985 at Anaheim, California.

2Both authors were supported in part by grants from the National Science Foundation.

©1986 American Mathematical Society

0002-9939/86 $1.00 + .25 per page

207
Theorem B gives the following two corollaries:

**COROLLARY C.** In case $K$ is the field of rational functions of $x$ with coefficients in $C$, if $\varphi$ and $\psi$ satisfy the hypotheses of Theorem B, then $\varphi'/\varphi = n(\psi'/\psi)$ is an algebraic function of $x$.

**COROLLARY D.** In case $K$ is the quotient field of $C\{x\}$ (the ring of convergent power series in $x$ with coefficients in $C$), if $\varphi$ and $\psi$ are formal power series in $x$ with coefficients in $C$, and if $\varphi$ and $\psi$ satisfy the hypotheses of Theorem B, then $\varphi$ and $\psi \in C\{x\}$ (i.e., $\varphi$ and $\psi$ are convergent).

The following examples illustrate these results.

**EXAMPLE 1.** The power series $\{\cos x\}^{1/n}$ and $J_{\nu}(x)^{1/n}$ ($n \geq 2$) do not satisfy any linear ordinary differential equation with polynomial coefficient (cf. also [1]).

**EXAMPLE 2.** The divergent power series $\{\sum_{m=0}^{\infty} m!x^m\}^{1/n}$ ($n \geq 2$) do not satisfy any linear ordinary differential equation with coefficients in $C\{x\}$.

2. **A fundamental lemma.** The following two important steps were utilized in our previous paper [2].

(1) Set

\[
\begin{align*}
P &= \sum_{h=0}^{m+1} p_h D^h, \\
Q &= \sum_{h=0}^{n+1} q_h D^h,
\end{align*}
\]

where $m$ and $n \in N$; $p_h$ and $q_h \in K$; in particular,

\[
\begin{align*}
p_{m+1} &\neq 0, \\
q_{n+1} &\neq 0.
\end{align*}
\]

For an element of $f$ of $K$, let us set

\[
\hat{f} = \sum_{h=0}^{\infty} \frac{f^{(h)}}{h!} x^h \in K[[x]],
\]

where $K[[x]]$ is the ring of formal power series in $x$ with coefficients in $K$. Then, $T(f) = \hat{f}$ defines an injective homomorphism of rings $T: K \rightarrow K[[x]]$ such that $T(f') = dT(f)/dx$. Corresponding to two operators $P$ and $Q$ of (2.1), let us consider two operators

\[
\begin{align*}
\hat{P} &= \sum_{h=0}^{m+1} \hat{p}_h (d/dx)^h, \\
\hat{Q} &= \sum_{h=0}^{n+1} \hat{q}_h (d/dx)^h.
\end{align*}
\]

We assume that $\varphi$ is an element of a differential field extension of $K$. Denote this extension by $\tilde{K}$. Then, $P(\varphi) = 0$ and $Q(1/\varphi) = 0$ imply, respectively, that the formal power series

\[
\hat{\varphi} = \sum_{h=0}^{\infty} \frac{\varphi^{(h)}}{h!} x^h \in \tilde{K}[[x]]
\]

satisfies $\hat{P}(\hat{\varphi}) = 0$ and $\hat{Q}(1/\hat{\varphi}) = 0$.

Observe that (2.2) implies $1/\hat{p}_{m+1} \in K[[x]]$ and $1/\hat{q}_{n+1} \in K[[x]]$. Therefore

\[
y = \frac{\hat{\varphi}}{\varphi} = 1 + \sum_{h=1}^{\infty} \frac{\varphi^{(h)}}{\varphi h!} x^h
\]
satisfies the differential equation

\[ y^{(m+1)} + \sum_{h=0}^{m} \frac{\hat{p}_h}{\hat{p}_{m+1}} y^{(h)} = 0, \]

and \( u = \varphi/\hat{\varphi} \) satisfies the differential equation

\[ u^{(n+1)} + \sum_{h=0}^{n} \frac{\hat{q}_h}{\hat{q}_{n+1}} u^{(h)} = 0. \]

In this manner, the general case was reduced to the case of formal power series.

2) Let \( k \) be a field. We denote by \( k[y_1, \ldots, y_p] \) the ring of polynomials in \( p \) indeterminates \( y_1, \ldots, y_p \) with coefficients in \( k \). For \( F \in k[y_1, \ldots, y_p] \) we set

\[ w(F) = \text{deg}_t F(\alpha_1 t, \alpha_2 t^2, \ldots, \alpha_p t^p), \]

where we regard \( F(\alpha_1 t, \alpha_2 t^2, \ldots, \alpha_p t^p) \) as a polynomial in \( t \) whose coefficients are polynomials in \( \alpha_1, \ldots, \alpha_p \). The following lemma was the fundamental algebraic tool in the proof of Theorem A.

**Lemma E.** Let \( n \) and \( p \) be positive integers, and let \( F_1, \ldots, F_p \in k[y_1, \ldots, y_p] \). Assume that

(i) \( w(F_p - y_p^n) < np \);
(ii) \( w(F_j) = nj \) \((j = 1, \ldots, p - 1)\);
(iii) \( w(F_j(y_1, \ldots, y_j, 0, \ldots, 0) - y_j^n) < nj \) \((j = 1, \ldots, p - 1)\).

Then, the system of algebraic equations

\[ F_j(y_1, \ldots, y_p) = 0 \quad (j = 1, \ldots, p) \]

admits only a finite number of solutions in any field extension of \( k \) and these solutions are algebraic over \( k \).

3. A lemma on differential equations. Let \( k \) be a field of characteristic zero, and let \( k[[x]] \) be the ring of formal power series in \( x \) with coefficients in \( k \). Then a linear ordinary differential equation

\[ y^{(m+1)} + \sum_{h=0}^{m} a_h(x) y^{(h)} = 0 \quad (a_h \in k[[x]]) \]

admits a canonical basis of \( m + 1 \) solutions of the form

\[ f_j = \frac{1}{j!} x^j + \sum_{h=m+1}^{\infty} f_{j,h} x^h \quad (j = 0, 1, \ldots, m), \]

where \( f_{j,h} \in k \).

Let \( k \) be a field extension of \( k \). If a formal power series

\[ \varphi = \sum_{h=0}^{m} c_h x^h \]

with coefficients in \( \tilde{k} \) satisfies (3.1), then we have

\[ \varphi = \sum_{j=0}^{m} (j!) c_j f_j. \]
Let us consider another linear ordinary differential equation:

\[(3.5)\quad u^{(p+1)} + \sum_{h=0}^{p} b_h(x) u^{(h)} = 0 \quad (b_h \in k[[x]]).\]

This equation also has a canonical basis of \(p + 1\) solutions of the form

\[(3.6)\quad g_j = \frac{1}{j!} x^j + \sum_{h=p+1}^{\infty} g_{j,h} x^h \quad (j = 0, 1, \ldots, p),\]

where \(g_{j,h} \in k\).

As in our previous paper [2], in order to prove Theorem B it suffices to prove the following result:

**Lemma F.** There are at most a finite number of solutions of the form

\[(3.7)\quad \begin{cases} y = f_0 + \sum_{j=1}^{m} \alpha_j f_j \quad (\alpha_j \in \tilde{k}), \\ u = g_0 + \sum_{j=1}^{p} \beta_j g_j \quad (\beta_j \in \tilde{k}) \end{cases}\]

of (3.1) and (3.5) respectively such that

\[(3.8)\quad y = u^n\]

if

\[(3.9)\quad n \geq m + 1.\]

For these solutions, the coefficients \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_p\) are algebraic over \(k\).

**Proof.** Set \(y = \sum_{h=0}^{\infty} \lambda_h x^h\) and \(u = \sum_{h=0}^{\infty} \mu_h x^h\), where \(\lambda_h\) and \(\mu_h \in \tilde{k}\). Then

\[(3.10)\quad \begin{cases} \lambda_0 = 1, & \mu_0 = 1, \\ \lambda_j = (1/j!)\alpha_j \quad (j = 1, \ldots, m), \\ \mu_j = (1/j!)\beta_j \quad (j = 1, \ldots, p), \\ \lambda_h = f_{0,h} + \sum_{j=1}^{m} (j!) f_{j,h} \lambda_j \quad (h \geq m + 1), \\ \mu_h = g_{0,h} + \sum_{j=1}^{p} (j!) g_{j,h} \mu_j \quad (h \geq p + 1). \end{cases}\]

In order that

\[(3.11)\quad \sum_{h=0}^{\infty} \lambda_h x^h = \left(\sum_{h=0}^{\infty} \mu_h x^h\right)^n,\]

it is necessary and sufficient that

\[(3.12)\quad \lambda_h = \sum_{h_1 + \cdots + h_n = h, h_j \geq 0} \mu_{h_1} \cdots \mu_{h_n} \quad \text{for } h \geq 0.\]

Let us define \(w(F)\) for \(F \in k[\mu_1, \ldots, \mu_p]\) in the same way as (2.6), and let us make the following observations:

(1) Utilizing the last formulas of (3.10), we regard \(\mu_h\) \((h \geq p + 1)\) as elements of \(k[\mu_1, \ldots, \mu_p]\). Then

\[(3.13)\quad w(\mu_h) \leq p < h \quad \text{for } h \geq p + 1.\]

(2) Utilizing \(\lambda_h = \sum \mu_{h_1} \cdots \mu_{h_n}\) for \(h = 1, \ldots, m\) (cf. (3.12)), we can regard \(\lambda_1, \ldots, \lambda_m\) as elements of \(k[\mu_1, \ldots, \mu_p]\). Then

\[(3.14)\quad w(\lambda_j) = j \quad (j = 1, \ldots, m).\]
(3) Utilizing the formulas for $\lambda_h$ ($h \geq m + 1$) of (3.10), we can regard $\lambda_h$ ($h \geq m + 1$) as elements of $k[\mu_1, \ldots, \mu_p]$. Then

$$w(\lambda_h) \leq m < h \quad \text{for } h \geq m + 1.$$  

Now, regarding (3.12) for $h \geq m + 1$ as a system of algebraic equations on $\mu_1, \ldots, \mu_p$, let us consider the system of $p$ equations

$$\lambda_h = \sum_{h_1+\cdots+h_n=h,h_j \geq 0} \mu_{h_1} \cdots \mu_{h_n} = 0$$

where $h = n, 2n, \ldots, pn$. Set

$$F_j(\mu_1, \ldots, \mu_p) = \lambda_{jn} - \sum_{h_1+\cdots+h_n=n_j} \mu_{h_1} \cdots \mu_{h_n} \quad (j = 1, \ldots, p).$$

Note that, if (3.9) is satisfied, we have

$$w(\lambda_{jn}) < n_j \quad (j = 1, \ldots, p)$$

(cf. (3.15)). Therefore,

$$w(F_j) < n_j \quad (j = 1, \ldots, p)$$

if (3.9) is satisfied. Furthermore, since

$$F_j(\mu_1, \ldots, \mu_j, 0, \ldots, 0) = \lambda_{jn} - \sum \mu_{h_1} \cdots \mu_{h_n}$$

if $\mu_{j+1} = 0, \ldots, \mu_p = 0$, where the sum is over all $n$-tuples $(h_1, \ldots, h_n)$ such that $h_1 + \cdots + h_n = n_j, h_j \geq 0$, it follows that

$$w(F_j(\mu_1, \ldots, \mu_j, 0, \ldots, 0) - \mu_j^n) < n_j \quad (j = 1, \ldots, p).$$

Thus applying Lemma E to (3.16) we can complete the proof of Lemma F, and thus Theorem B.

Using different methods, our results have been substantially generalized in [3].

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90089-1113

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455