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ON THE CLIFFORD INDEX OF ALGEBRAIC CURVES

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ABSTRACT. Here we prove (over $\mathbf{C}$) that a general $(e+2)$-gonal algebraic curve of genus p has no g^r_d with $d \leq p-1$, $r \geq 2$ and $d - 2r \leq e$.

In this note we give the expected answer (yes) to a conjecture raised in [**2**, Conjecture 3.8]. The proof uses only the more elementary part of a theory introduced by D. Eisenbud and J. Harris in [**4**].

Let X be a smooth, connected, complete curve and L a line bundle on X. The Clifford index $\mathrm{Cliff}(L)$ of L is defined by $\mathrm{Cliff}(L) = \deg L - 2(h^0(X,L) - 1)$. The Clifford index $\mathrm{Cliff}(X)$ of X is defined by $\mathrm{Cliff}(X) = \min\{\mathrm{Cliff}(L): L$ is a line bundle on X with $h^0(X,L) \geq 2$ and $h^1(X,L) \geq 2\}$. If X has genus p in the definition of $\mathrm{Cliff}(X)$ we may use the condition "$\deg L \leq p-1$" instead of the condition "$h^1(X,L) \geq 2$".

Fix an algebraically closed field k with $ch(k) = 0$. Here we prove the following result:

THEOREM. *A general $(e+2)$-gonal curve X of genus $p \geq 2e+2$ has* $\mathrm{Cliff}(X) = e$. *Furthermore X has no g^r_{e+2r} with $r \geq 2$ and $e + 2r \leq p-1$.*

PROOF. We use induction on the genus. By Brill and Noether's theory [**3**] the induction starts for example at the genus $2e+2$, when a curve with general moduli has a g^1_{e+2}. Assume the result for a general $(e+2)$-gonal curve of genus $p-1$. Fix integers r, d with $r \geq 2$, $d \leq p-1$ and $d - 2r \leq e$. Let P be a point of A at which a g^1_{e+2} on A ramifies. Let E be an elliptic curve and $0 \in E$. Let Y be the union of A and E with the points P and 0 identified. By the theory of admissible covers by J. Harris and D. Mumford [**5**, Proof of Theorem 5.6, Corollary 4, p. 71], Y is in the closure in the moduli space of stable curves of genus p of the set of smooth $(e+2)$-gonal curves. Let B be a discrete valuation ring with closed point o and generic point t. Let $f: Z \to B$ be flat and proper with Z smooth, $f^{-1}(o) \cong Y$ and with $f^{-1}(t)$ an $(e+2)$-gonal smooth curve of genus p. By Lefschetz principle and the existence of $\mathcal{G}^r_d$ on a suitable cover of the moduli space $\mathcal{M}_p$, to obtain a contradiction we may assume that the geometric general fiber of f (i.e. the extension of $f^{-1}(t)$ over the algebraic closure of $k(t)$) has a g^r_d. Certainly this g^r_d is defined over a finite extension of $k(t)$. As in [**4**, §2] we find $j \geq 0$ and a g^r_d limit on a semistable curve Y' defined over k, Y' union of A, a chain of rational curves D_i, $1 \leq i \leq j$, and E, with $A \cap D_i = \varnothing$ if $i > 1$, $A \cap D_1 = \{P\}$, $D_i \cap D_s \neq \varnothing$ if and only if $|i - j| \leq 1$, $\mathrm{card}(D_i \cap D_{i+1}) = 1$ for $1 \leq i \leq j-1$, $E \cap D_s = \varnothing$ if $s < j$, $E \cap D_j = \{0\}$, $E \cap A = \varnothing$ (unless $j = 0$). If $j = 0$, set $Y' := Y$. By definition of a g^r_d limit,

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its existence implies the existence of a g_d^r on A. If $d \leq p-2$, this contradicts the inductive assumption and the choice of A. If $d = p-1$, we may assume that the associated line bundle L on A has no base point. By Riemann and Roch we have $h^1(A, L) = r-1$, and if $r \geq 3$, we obtain a contradiction by duality and the inductive assumption. Hence we may assume $r = 2$, $d = p-1$ and L base-point free. We obtain $p-1-4 \leq e$ and by the choice of $p > 2e+2$ we find $e \leq 2$. The only cases to check are the cases with $e = 0$ or $e = 1$, $g = 6$ or $e = 2$, and $g = 7$, which are well known. If $e = 0$, A is hyperelliptic. If $g = 6$, a general 3-gonal curve has no g_5^2 (necessarily not composed with a pencil) by a dimensional count. If $g = 7$, a general 4-gonal curve A cannot have a g_6^2 which maps A birationally onto $C \subset \mathbf{P}^2$, $\deg C = 6$, because C can have only double points (A has no g_3^1), hence it has at least 3 g_4^1, contradicting a theorem of E. Arbarello and M. Cornalba [**1**, Theorem 2.6]. If A has a g_6^2 which maps A nonbirationally onto $C \subset \mathbf{P}^2$, C must be elliptic and A elliptic-hyperelliptic; again A cannot have infinite g_4^1. $\square$

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