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ON THE OSCILLATION OF ALMOST-PERIODIC STURM-LIOUVILLE OPERATORS WITH AN ARBITRARY COUPLING CONSTANT[1]

S. G. HALVORSEN AND A. B. MINGARELLI[2]

ABSTRACT. In this paper we characterize those (Bohr) almost periodic functions V on $\mathbf{R}$ for which the Sturm-Liouville equations

$$-y'' + \lambda V(x)y = 0, \qquad x \in \mathbf{R},$$

are oscillatory at $\pm\infty$ for every real $\lambda \neq 0$, or, equivalently, for which there exists a real $\lambda \neq 0$ such that the equation has a positive solution on $\mathbf{R}$.

1. Introduction. In the study of the disconjugacy domain $\mathcal{D}$ (see [**4**]) of a linear second order differential equation with two parameters α, β,

$$-y'' + (\alpha - \beta V(x))y = 0, \qquad x \in \mathbf{R}, \tag{1.1}$$

one is inevitably drawn into the important case when $\mathcal{D} \subseteq \{(\alpha,\beta)\colon \alpha > 0\} \cup \{(0,0)\}$ which occurs, for example, when V is periodic and has mean value equal to zero, as in Hill's equation. We recall that the equation

$$-y'' + V(x)y = 0, \qquad x \in \mathbf{R},$$

is said to be *disconjugate* on $\mathbf{R}$ provided every one of its (nonidentically zero) solutions has at most one zero in $(-\infty, \infty)$. This is equivalent to the fact that there exists a solution $y(x) > 0$ for $x \in \mathbf{R}$. $\mathcal{D}$ is then the collection of $(\alpha, \beta) \in \mathbf{R}^2$ for which (1.1) is disconjugate on $\mathbf{R}$. If $\mathcal{D}$ is contained in the right-half plane of the parameter space $\mathbf{R}^2$, as above, it follows that

$$-y'' + \lambda V(x)y = 0 \tag{1.2}$$

is oscillatory (at both ends $\pm\infty$) for *every* real $\lambda \neq 0$. General conditions on V for which this behavior is realized may be found in our recent monograph [**3**]. The case when V is a (Bohr) almost periodic function was considered by Markus and Moore [**4**]. In this case we show that it is possible to characterize those almost-periodic V for which (1.2) is oscillatory at $\pm\infty$ for every real $\lambda \neq 0$. The results used to obtain this characterization are drawn from oscillation theory, in particular, results of Moore [**5**] and Wintner [**7**] are central to our investigations. A by-product of our techniques is that various classes of generalized almost-periodic functions such as those considered by Weyl and Besicovitch (see [**1**] for more details) can also be treated.

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2. Basic results and terminology. In the sequel, $M\{V(x)\}$ will denote the *mean-value* of an almost periodic function,

$$M\{V(x)\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T V(s)\,ds.$$

We recall that a.p. functions have a mean value that is uniform, in the sense that

$$M\{V(x)\} = \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} V(s)\,ds$$

uniformly for $a \in \mathbf{R}$ (see [**1**, **2**]). For a given V, v will denote some, generally unspecified, indefinite integral of V.

3. The main theorem.

THEOREM 3.1. *Let $V \not\equiv 0$ be an almost periodic function. Then a necessary and sufficient condition for* (1.2) *to be oscillatory at $\pm\infty$ for every real $\lambda \neq 0$ is that $M\{V(x)\} = 0$.*

COROLLARY 3.2. *The following statements are equivalent:*
(i) *The equation* (1.2) *is oscillatory at $\pm\infty$ for every real $\lambda \neq 0$.*
(ii) *These exist finite numbers $\lambda^+ > 0$ and $\lambda^- < 0$, such that* (1.2) *is oscillatory at $\pm\infty$ for every $\lambda \in (\lambda^-, \lambda^+)$, $\lambda \neq 0$.*
(iii) $M\{V(x)\} = 0$.

PROOF. That (i)$\Leftrightarrow$(iii) and (i)$\Rightarrow$(ii) is clear. That (ii)$\Rightarrow$(i) follows from the convexity of the disconjugacy domain [**4**].

The case V purely periodic with $M\{V(x)\} = 0$ of Theorem 3.1 can be found in Staněk [**6**]. However this case actually follows directly from [**4**, Theorems 2 and 6].

PROOF OF THEOREM 3.1 (*Sufficiency*). This was essentially shown in [**4**, Theorem 2]. Another proof may also be found in W. Coppel's monograph *Disconjugacy* [Theorem 14].

(*Necessity*). We will show that whenever $M\{V(x)\} \neq 0$, there exists a value of $\lambda \in \mathbf{R}$, $\lambda \neq 0$, for which (1.2) is disconjugate on $\mathbf{R}$. It will follow from this that (1.2) will be oscillatory for every real $\lambda \neq 0$, only if $M\{V(x)\} = 0$.

To this end let $M\{V(x)\} = m \neq 0$ and consider the single differential equation in the two real parameters μ, ν:

$$y'' + (-\nu + \mu V(x))y = 0 \tag{3.1}$$

on $[0,\infty)$. (Note that nonoscillation on $[0,\infty)$ implies disconjugacy on $[0,\infty)$ and so on $(-\infty,\infty)$ by results in [**4**].) Then (3.1) may be rewritten as

$$y'' + (-\alpha + \beta V^*(x))y = 0, \tag{3.2}$$

where $\beta = \mu$, $\alpha = \nu - m\mu$ and $M\{V^*(x)\} = 0$. (Let $\alpha > 0, \beta \neq 0$.) We now make the transformation $y = z\exp(-x\sqrt{\alpha})$ and $t = (1/2\sqrt{\alpha})\exp(2\sqrt{\alpha}x)$. This leads us to the equation

$$z'' + \beta e^{-4\sqrt{\alpha}x} V(x) z = 0$$

and $f(t) = \beta V(x)\exp(-4\sqrt{\alpha}x)$. The x-interval $(-\infty,\infty)$ goes into the half-axis, $[0,\infty)$. Now, (rewriting V for V^*),

(3.3)

$$\begin{aligned} t\int_t^\infty f(x)\,ds &= \frac{\beta e^{2\sqrt{\alpha}x}}{2\sqrt{\alpha}}\int_x^\infty V(x)\exp(-2\sqrt{\alpha}s)\,ds \\ &= \frac{\beta}{2\sqrt{\alpha}}\int_0^\infty e^{-2\sqrt{\alpha}\tau}V(x+\tau)\,d\tau = \beta\int_0^\infty e^{-2\sqrt{\alpha}\tau}\int_0^\tau V(x+s)\,ds\,d\tau \\ &= \beta\int_0^\infty \tau e^{-2\sqrt{\alpha}\tau}\left[\frac{1}{\tau}\int_0^\tau V(x+s)\,ds\right]d\tau \\ &= \beta\int_0^\infty \tau e^{-2\sqrt{\alpha}\tau}\left[\frac{1}{\tau}\int_x^{x+\tau} V(s)\,ds\right]d\tau. \end{aligned}$$

(note that both the first two integrals converge since V is bounded). Since $M\{V(x)\} = 0$ and V is a.p., then for every $\varepsilon > 0$ there exists $\tau_0(\varepsilon) > 0$ for which

$$\sup_{x\in\mathbf{R}}\left|\frac{1}{\tau}\int_x^{x+\tau} V(s)\,ds\right| \le \varepsilon \tag{3.4}$$

for $\tau \ge \tau_0$ [**2**, p. 44]. Thus let $T > 0$ and rewrite (3.3) as an integral over $[0,T]$ plus an integral over $[T,\infty)$. Then

$$\left|\int_0^\infty \tau e^{-2\sqrt{\alpha}\tau}\left[\frac{1}{\tau}\int_x^{x+\tau} V(s)\,ds\right]d\tau\right| \le M\int_0^T \tau e^{-2\sqrt{\alpha}\tau}d\tau \tag{3.5}$$

as it is certainly the case that the integral appearing in the square parentheses is bounded, by $M = M(T)$ say, as it is a continuous function of $\tau \in [0,T]$. (Note that $\sup\{M(T): T \ge 0\} < \infty$ on account of (3.4).) Moreover, since $M\{V(x)\} = 0$ we have

$$\left|\int_T^\infty \tau e^{-2\sqrt{\alpha}\tau}\left[\frac{1}{\tau}\int_x^{x+\tau} V(s)\,ds\right]d\tau\right| \tag{3.6}$$

$$\le \sup_{\tau\in[T,\infty)}\left|\frac{1}{\tau}\int_x^{x+\tau} V(s)\,ds\right|\cdot\int_T^\infty \tau e^{-2\sqrt{\alpha}\tau}d\tau$$

$$\le \varepsilon(T)\int_T^\infty \tau e^{-2\sqrt{\alpha}\tau}d\tau. \tag{3.7}$$

Combining the estimates (3.5), (3.7) and writing $K = 2\sqrt{\alpha}T$ we obtain

$$\begin{aligned} \left|t\int_t^\infty f(s)\,ds\right| &\le \frac{M|\beta|}{4\alpha}[1-(K+1)e^{-K}] + \frac{\varepsilon|\beta|}{4\alpha}(K+1)e^{-K} \\ &\le \frac{|\beta|}{4\alpha}\left\{\frac{M}{2}K^2 + \varepsilon(T)\right\}. \end{aligned} \tag{3.8}$$

We may now let $T \to \infty$ in such a way that $T = O(\alpha^{-1/2})$ as $\alpha \to 0^+$. Then, the uniformity of the mean value (3.4) will imply that $\varepsilon(t) \to 0$ uniformly in x (see (3.6)). Moreover we will also have $K \to 0$ (as $\alpha \to 0^+$).

Hence if $\alpha > 0$ is sufficiently small we see that

$$\frac{|\beta|}{4\alpha}\left\{\frac{M}{2}K^2 + \varepsilon(T)\right\} \le \frac{1}{4}, \tag{3.9}$$

i.e., if $|\beta| \leq \alpha\Psi(\alpha)$, where $\Psi(\alpha) = \{MK^2/2 + \varepsilon(T)\}^{-1}$ for an appropriately large T which we then fix, then (3.2) will be nonoscillatory (and so disconjugate) on $[0, \infty)$ on account of [**5**, Theorem 6; **7**], i.e., (3.2) will be disconjugate on $(-\infty, \infty)$. It follows from (3.10) that the disconjugacy domain just touches the β-axis at the origin, and at $(0, 0)$ we have a vertical tangent!

We now return to (3.2). Assume $m > 0$. We set $\nu = 0$ in (3.1), i.e., $\alpha = -m\mu = -m\beta$ in (3.2). Then from the preceding discussion it follows that the line $\alpha + m\beta = 0$ must intersect the disconjugacy domain of (3.2) for some $\alpha > 0$ and some range of negative β's, say, $0 > \beta \geq \beta_0$. Similarly if $m < 0$, we may find such a range of positive β's, $0 < \beta \leq \beta_1$. In either case there exists $\mu \neq 0$ for which (3.1) (with $\nu = 0$) is disconjugate on $\mathbf{R}$. This completes the proof of the necessity and of the theorem.

REMARK. The proof of the necessity shows that the disconjugacy domain of an equation (3.2) with V^* a.p. and $M\{V^*(x)\} = 0$ has a (boundary with a) vertical tangent at $(0, 0)$ and lies completely in the right half-plane $\{\alpha > 0\} \cup \{(0, 0)\}$. This extends a correpsonding result of Markus and Moore [**4**, p. 106, Theorem 6] wherein it is further assumed that $v(x)$ (defined earlier) is also a.p. Furthermore the necessity merely required the uniformity of the mean-value of V and consequently holds for potentials which may not be a.p. For example Stepanoff, Weyl/Besicovitch a.p. functions inherit this property as well as many other (nongeneralized a.p.) functions.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TRONDHEIM, N.T.H. 7034 TRONDHEIM, NORWAY

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO K1N 6N5, CANADA