A NEW FIXED POINT THEOREM ON
DEMI-COMPACT 1-SET-CONTRACTION MAPPINGS
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ABSTRACT. W. V. Petryshyn studied the fixed point theorem of demi-
compact mappings. This paper gives a new fixed point theorem of demi-
compact 1-set-contraction mappings in the norm form.

W. V. Petryshyn [1,2] studied the fixed point theorem of demi-compact mapp-
ings. This paper gives a new fixed point theorem on demi-compact 1-set-contraction mappings. Let \( P \) be a cone of a real Banach space \( E \). Set

\[
\begin{align*}
P_r &= \{ x \in P \mid \|x\| < r \}, & \partial P_r &= \{ x \in P \mid \|x\| = r \}, \\
\overline{P}_r &= \{ x \in P \mid r < \|x\| < R \}, & \overline{P}_{r,R} &= \{ x \in P \mid r \leq \|x\| \leq R \},
\end{align*}
\]

First we state a result obtained by A. J. Potter [3], R. D. Nussbaum [4], and P. M. Fitzpatrick and W. V. Petryshyn [5].

**Lemma 1.** Let \( A : \overline{P}_r \rightarrow P \) be a condensing mapping. Suppose that \( A \) satisfies

1. There is some \( w > \theta \) such that \( x - Ax \geq Xw \) for any \( X > 0 \) and \( x \in \partial P_r \),
2. \( \lambda x \neq Ax \) for \( \lambda > 1 \) and \( x \in \partial P_R \).

Then \( A \) has a fixed point \( x^* \in \overline{P}_{r,R} \).

**Remark.** Lemma 1 is a particular condition of Theorem 3.2 of [5] and Remark 1.2 of [4].

**Corollary [4].** Let \( A : \overline{P}_r \rightarrow P \) be condensing. Suppose that \( A \) satisfies one of the following conditions:

1. \( x \in \partial P_r \Rightarrow Ax \geq x \) and \( x \in \partial P_r \Rightarrow Ax \leq x \),
2. \( x \in \partial P_r \Rightarrow Ax \geq x \) and \( x \in \partial P_R \Rightarrow Ax \leq x \).

Then \( A \) has a fixed point \( x^* \in \overline{P}_{r,R} \).

Obviously, suppose that condition (1)' is satisfied. Then

1. \( x \in \partial P_r, \lambda \geq 1 \Rightarrow Ax \neq \lambda x \) (if \( Ax = \lambda x \) then \( Ax = \lambda x \geq x \)).
2. Let any \( h > \theta \). Then \( x \in \partial P_r, \lambda \geq 0 \Rightarrow Ax \neq \lambda h \) (if \( x - Ax = \lambda h \), then \( x - Ax \geq \theta \), i.e. \( Ax \leq x \)). So \( A \) has a fixed point \( x^* \in \overline{P}_{r,R} \).

Analogously, we can prove that \( A \) has a fixed point \( x^* \in \overline{P}_{r,R} \) under (2)'.

**Lemma 2.** Let \( P \) be a cone of a real Banach space \( E \), and let the norm \( \|x\| \) be increasing with respect to \( P \). Suppose that \( A : \overline{P}_r \rightarrow P \) is a \( k \)-set-contraction mapping \( (0 < k < 1) \), which satisfies one of the following conditions:

\[
\begin{align*}
(H_1) & \quad x \in \partial P_r \Rightarrow \|Ax\| \leq \|x\|, & x \in \partial P_R \Rightarrow \|Ax\| \geq \|x\|,
\end{align*}
\]

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or

\[(H_2) \quad x \in \partial P_R \Rightarrow \|Ax\| \leq \|x\|; \quad x \in \partial P_r \Rightarrow \|Ax\| \geq \|x\|.\]

Then A has a fixed point in \(P_{r,R}\).

**PROOF.** We prove only that Lemma 1 holds under \((H_1)\). A is a strict-set-contraction.

We set the operator as follows:

\[
\tilde{A}_n x = \begin{cases} 
1 + \frac{\|x\| - s}{n(R - s)} A x, & x \in P, s \leq \|x\| < R, \\
1 - \frac{s - \|x\|}{n(s - r)} A x, & x \in P, r \leq \|x\| < s,
\end{cases}
\]

where \(s = \frac{1}{2}(r + R)\). \(\tilde{A}_n\) is a continuous and bounded operator.

We discuss the operator

\[
B_n x = \left(\frac{\|x\|}{nK}\right) A x, \quad x \in \overline{P}_{r,R}, K = \min\{R - s, s - r\}.
\]

Let \(\Omega\) be a open set in \(\overline{P}_{r,R}\), by definition and property of the measure \(\alpha(\Omega)\) of noncompactness of \(\Omega\),

\[
\alpha(\Omega) = \inf \left\{ d | s_i \subset \Omega, \bigcup_{i=1}^{N} s_i = \Omega, \text{diam } s_i < d \right\},
\]

\[
\alpha(\Omega) \leq \alpha(\overline{P}_{r,R}), \quad B_n(s_i) \subset B_n(\Omega), \quad \bigcup_{i=1}^{N} B_n(s_i) = B_n(\Omega),
\]

\[
\text{diam } B_n(s_i) \leq \frac{(2R/nK)}{\text{diam } A(s_i)}, \quad \alpha(B_n(\Omega)) \leq \frac{(2R/nK)\alpha(A(\Omega))}{\text{diam } A(\Omega)} \leq (2R/nK)k \alpha(\Omega).
\]

Hence \(B_n\) is a \((2R/nK)k\)-set-contraction mapping, as \(n \to \infty\), \((2R/nK)k \to 0\). So when \(n\) is large enough, \(B_n\) is a strict-set-contraction mapping on \(\overline{P}_{r,R}\).

We set operators

\[
A_n^{(1)} x = \left(1 + \frac{\|x\| - s}{n(R - s)}\right) A x, \quad x \in P, s \leq \|x\| \leq R,
\]

\[
A_n^{(2)} x = \left(1 - \frac{s - \|x\|}{n(s - r)}\right) A x, \quad x \in P, r \leq \|x\| < s.
\]

Obviously \(A_n^{(1)}\) and \(A_n^{(2)}\) are \(k\)-set-contraction mappings, where

\[
\tilde{k} = \left(1 + \frac{2R + s}{nK}\right) k.
\]

When \(n\) is large enough, \(0 < \tilde{k} < 1\). Hence \(A_n^{(1)}\) and \(A_n^{(2)}\) are strict-set-contraction mappings for large enough \(n\).

Set

\[
P_1 = \{x | x \in P, s \leq \|x\| \leq R\}, \quad P_2 = \{x | x \in P, r \leq \|x\| < s\}, \quad \overline{P}_{r,R} = P_1 \cup P_2.
\]
Hence,
\[
\Omega = (\Omega \cap P_1) \cup (\Omega \cap P_2),
\]
\[
\tilde{A}_n(\Omega) = [\tilde{A}_n(\Omega \cap P_1)] \cup [\tilde{A}_n(\Omega \cap P_2)]
\]
\[
= [A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)],
\]
\[
\alpha(A_n(\Omega)) = \alpha([A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)])
\]
\[
= \max\{\alpha[A_n^{(1)}(\Omega \cap P_1)], \alpha[A_n^{(2)}(\Omega \cap P_2)]\}
\]
\[
\leq \kappa \alpha(\Omega), \quad 0 < \kappa < 1.
\]
Then \(\tilde{A}_n\) is a strict-set-contraction mapping. When \(n\) is large enough, set
\[
A_n x = \begin{cases} 
\tilde{A}_n x, & r \leq \|x\| \leq R, \\
Ax, & 0 \leq \|x\| < r.
\end{cases}
\]
\(A_n\) satisfies a condition of the Corollary of Lemma 1.

In fact, if there is an \(x_0 \in \partial P_r\) such that \(A_n x_0 \geq x_0\), since norm is monotonic increasing and, by condition (H1),
\[
r = \|x_0\| \leq \|A_n x_0\| = (1 - 1/n)\|x_0\| \leq (1 - 1/n)\|x_0\| = (1 - 1/n)r < r.
\]
This is contradiction. Then \(x \in \partial P_r, A_n x \geq x\).
If there is an \(x_1 \in \partial P_r\) such that \(A_n x_1 \leq x_1\), we obtain
\[
R = \|x_1\| \geq \|A_n x_1\| = (1 + 1/n)\|A_x\| \geq (1 + 1/n)\|x\| = (1 + 1/n)R > R.
\]
This is also contradiction. Then \(x \in \partial P_R, A_n x \leq x\).

Hence \(A_n\) satisfies a condition of the Corollary of Lemma 1, and there exists \(x_n^* \in \overline{P}_{r,R}\) such that \(A_n x_n^* = x_n^*\).
If \(P_1\) includes, without loss of generality, subsequence \(\{x_{n_k}^*\}\) of \(\{x_n^*\}\),

\[
x_{n_k}^* = A_{n_k} x_{n_k}^* = \left(1 + \frac{\|x_{n_k}^*\| - s}{n_k(R - s)}\right) Ax_{n_k}^*.
\]

Since \(A\) is a strict-set-contraction mapping, the set \(\{\|Ax_{n_k}^*\|\}\) must be bounded.
\[
\|x_{n_k}^* - Ax_{n_k}^*\| = \frac{\|x_{n_k}^*\| - s}{n_k(R - s)} \|Ax_{n_k}^*\| < \frac{1}{n_k} \|Ax_{n_k}^*\| \to 0 \quad (n_k \to \infty),
\]
i.e., \(x_{n_k}^* - Ax_{n_k}^* \to \theta (n_k \to \infty)\).

Since a strict-set-contraction mapping is a demi-compact 1-set-contraction, then there exists a convergent subsequence of \(\{x_{n_k}^*\}\), which we write down also as \(\{x_{n_k}^*\}\), and \(x_{n_k}^* \to x^* (n_k \to \infty)\). Since \(\overline{P}_{r,R}\) is closed, then \(x^* \in \overline{P}_{r,R}\). By (1), as \(n_k \to \infty\), we imply that \(Ax^* = x^*\). This proves that \(A\) has a fixed point \(x^* \in \overline{P}_{r,R}\) under condition (H1).

Analogously, we can prove that \(A\) has a fixed point \(x^* \in \overline{P}_{r,R}\) under condition (H2). Q.E.D.

We obtain the following main theorem.

**THEOREM.** Let \(P\) be a cone of a real Banach space \(E\), and let norm be increasing with respect to \(P\). Suppose that \(A: \overline{P}_R \to P\) demi-compact 1-set-contraction mapping and there is a \(\delta > 0\) such that

\[
(i) \quad x \in \partial P_r \Rightarrow \|Ax\| \leq \|x\|, \quad x \in \partial P_R \Rightarrow \|Ax\| \geq (1 + \delta)\|x\|,
\]

\[
(ii) \quad x \in \partial P_r \Rightarrow \|Ax\| \leq \|x\|, \quad x \in \partial P_R \Rightarrow \|Ax\| \geq (1 + \delta)\|x\|,
\]

\[
(iii) \quad x \in \partial P_r \Rightarrow \|Ax\| \leq \|x\|, \quad x \in \partial P_R \Rightarrow \|Ax\| \geq (1 + \delta)\|x\|,
\]

\[
(iv) \quad x \in \partial P_r \Rightarrow \|Ax\| \leq \|x\|, \quad x \in \partial P_R \Rightarrow \|Ax\| \geq (1 + \delta)\|x\|.
\]
or

(ii) \( x \in \partial P \Rightarrow \|Ax\| \leq \|x\|, \ x \in \partial P \Rightarrow \|Ax\| \geq (1 + \delta)\|x\| \).

Then \( A \) has a fixed point in \( \overline{P}_{r,R} \).

PROOF. Since \( A: \overline{P}_R \rightarrow P \) is a 1-set-contraction mapping, for an arbitrary open subset \( \Omega \subset \overline{P}_{r,R}, \ \alpha(A(\Omega)) \leq \alpha(\Omega) \).

We construct a new operator as follows:

\[ A_n x = \lambda_n Ax, \quad \lambda_n = (n - 1)/n. \]

Hence,

\[ \alpha(A_n(\Omega)) = \alpha(\lambda_n A(\Omega)) = \lambda_n \alpha(A(\Omega)) \leq \lambda_n \alpha(\Omega). \]

Then \( A_n \) is a strict-set-contraction mapping.

If condition (i) holds when \( n \) is large enough such that \( 1 > \lambda_n > 1/(1 + \delta) \), then

\[ x \in \partial P, \quad \|A_n x\| = \lambda_n \|Ax\| \leq \|x\|, \]
\[ x \in \partial P, \quad \|A_n x\| = \lambda_n \|Ax\| \geq \|Ax\|/(1 + \delta) \geq \|x\|. \]

Hence \( A_n \) satisfies a condition of Lemma 2, and there exist \( x_n \in \overline{P}_{r,R} \) such that \( A_n x_n = x_n \), i.e. \( \lambda_n Ax_n = x_n \),

\[ x_n - Ax_n = x_n - \lambda_n Ax_n + \lambda_n Ax_n - Ax_n = (\lambda_n - 1)Ax_n \rightarrow \theta \quad (n \rightarrow \infty). \]

Because \( A \) is a demi-compact mapping, there exists a convergent subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \rightarrow x_0 \in \overline{P}_{r,R} \). By continuity of \( A \), \( x_0 = Ax_0 \).

Analogously, we can prove that \( A \) has a fixed point \( x^* \in \overline{P}_{r,R} \) under condition (ii).

REFERENCES


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