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SPECTRAL CUTTING FOR A CLASS OF SUBNORMAL OPERATORS

BHUSHAN L. WADHWA

ABSTRACT. Let S be a subnormal decomposable operator on a Hilbert space $\mathcal{H}$. (The dual of the Bergman shift belongs to this class.) It is shown that for any closed set δ with nonempty intersection with the spectrum of S, $\mathcal{H}$ can be decomposed as $M(\delta)\oplus\overline{M(\delta')}\oplus M^*$, where $M(\delta)$ and $\overline{M(\delta')}$ are hyperinvariant under T, and M^* is invariant under T^* with spectrum of $T^*|M^*$ contained in the conjugate of the boundary of δ. The minimal normal extension of the subnormal operator $T|M(\delta)$ is also considered.

An operator S on a Hilbert space $\mathcal{H}$ is called subnormal in case it is the restriction of a normal operator N acting on a superspace $\mathcal{K} \supset \mathcal{H}$ such that $N\mathcal{H} \subset \mathcal{H}$. If the normal operator N is chosen to be minimal in the sense that

$$\mathcal{K} = \operatorname{span}\{N^{*j}h\colon h \in \mathcal{H},\ j = 0, 1, 2, \ldots\},$$

then the operator N is uniquely determined up to a unitary equivalence which leaves S and $\mathcal{H}$ fixed. Relative to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, the operator N has the 2×2 matrix representation

$$N = \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix}. \tag{1.1}$$

The operator N is called the minimal normal extension (m.n.e.) of S. If S is pure (no reducing subspace on which S is normal), the operator T in the above representation is subnormal (the m.n.e. of T is N^*) and is called the dual of S.

For a recent account of the theory of subnormal operators see Conway [**5**].

Let $L^2(\Delta)$ denote the usual Hilbert space of functions on Δ, the open unit disc, which are square integrable with respect to the Lebesgue planar measure. Let $L^2_a(\Delta)$ denote the closure in $L^2(\Delta)$ of functions in $L^2(\Delta)$ which are analytic in Δ. The normal operator N defined on $L^2(\Delta)$ by $Nf(z) = zf(z)$ leaves $L^2_a(\Delta)$ invariant, the subnormal operator obtained by restricting N to $L^2_a(\Delta)$ is called the Bergman shift. The orthogonal complement of the space $L^2_a(\Delta)$ in $L^2(\Delta)$ can be identified with the Sobolev space $w_0^{1,2}(\Delta)$ and the operator T defined on $w_0^{1,2}(\Delta)$ by $Tf(z) = \bar{z}f(z)$ is a subnormal operator with N^* on $L^2(\Delta)$ as its m.n.e. The spectrum of T is the same as its essential spectrum and it is the closed unit disc. This operator T, the dual of the Bergman shift, is a generalized scalar operator (Apostol, Foiaş and Voiculescu [**1**]), which means that T has $C^\infty(\Delta)$-functional calculus. The reader is referred to Conway [**6**] for more details of this fascinating operator.

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Let B be a bounded operator on a Hilbert space $\mathcal{H}$. A subspace M of $\mathcal{H}$ is called a maximal spectral space of B if M is invariant under B and $Z \subset M$ for every subspace Z which is invariant under B and $\sigma(B|Z) \subset \sigma(B|M)$. The operator B is called a decomposable operator if for every finite open covering $\{G_1, G_2, \ldots, G_n\}$ of the spectrum of B, $\sigma(B)$, there exist maximal spectral subspaces $M_1, M_2, \ldots, M_n$ such that $\mathcal{H} = M_1 + M_2 + \cdots + M_n$ and the $\sigma(B|M_i)$ is contained in G_i for $i = 1, \ldots, n$. The notion of a decomposable operator is due to Foiaş [**9**]. One can easily show that a generalized scalar operator B is a decomposable operator [**4**] by taking $\{\phi_1, \phi_2, \ldots, \phi_n\}$ as a C^∞-partition of unity subordinate to a given covering $\{G_1, G_2, \ldots, G_n\}$ of the $\sigma(B)$; the corresponding maximal spectral space is precisely the closure of range of $\mathcal{U}\phi_i(z)$, where $\mathcal{U}$ is the mapping which induces the C^∞-functional calculus of the operator B.

The dual of the Bergman shift is a decomposable operator which is subnormal. Another equally fascinating subnormal decomposable operator can be obtained from McKissick's example of a normal function algebra (see Stout [**13**, pp. 345–355]). These operators being decomposable have an abundance of maximal spectral (and thus hyperinvariant) subspaces. We shall get a finer decomposition of the underlying space from the fact that these operators are also subnormal. Let us introduce some more terminology.

Let B be a bounded operator on the Hilbert space $\mathcal{H}$. For a fixed vector x in $\mathcal{H}$, the local resolvent is the (multiple valued) function $x_B(\lambda)$ which consists of all possible analytic continuations of $(B-\lambda)^{-1}x$ in the complex plane from the resolvent set $\rho(B)$. The operator B is said to have the single valued extension property (s.v.e.p.) in case $x_B(\lambda)$ is single valued for every $x \in \mathcal{H}$. In case B has the s.v.e.p., the maximal domain of $x_B(\lambda)$, $\rho(B;x)$, is called the local resolvent set of x and $\sigma(B;x) =$ the complement of $\rho(B;x)$, the local spectrum of x. For a closed set δ in C, $M(B;\delta) = \{x \in \mathcal{H} : \sigma(B;x) \subset \delta\}$ is a linear manifold invariant under B. In case B is a decomposable operator, then B has the s.v.e.p.; $M(B;\delta)$ is closed, $M(B;\delta)$ is a maximal spectral space of B. Furthermore all maximal spectral spaces of B are of this type (if M is a maximal spectral space of B, then $M = M(B;\sigma(B|M))$). For further details the reader is referred to Colojoara and Foiaş [**4**], Dunford and Schwartz [**8**] and Clancey [**2**].

In this note we shall study subnormal operators which are decomposable. These operators, like normal operators, have an abundance of hyperinvariant subspaces. In the first section we shall show that for any open set G of the complex plane, $\mathcal{H} = M(S;G') \oplus \overline{M(S;G)} \oplus M^*$, where M^* is invariant under S^* and the $\sigma(S^*|M^*) \subset \partial\tilde{G}$ (complex conjugate of the ∂G). In the second section we shall consider the question of the m.n.e. of $S|\overline{M(S;G)}$ and that of $S|M(S;G')$. G' denotes the complement of G.

1. Spectral decomposition. The following proposition about normal operators is well known and is included to serve as a motivation for the main theorem of this section.

PROPOSITION 1.1. *Let $N = \int \lambda\, dE(\lambda)$ be a normal operator acting on a Hilbert space $\mathcal{K}$. If G is an open set in C, then $\mathcal{K} = \overline{M(N;G)} \oplus M(N;G')$.*

PROOF. If $\sigma(N;x) \cap \sigma(N;y) = \varnothing$, then, from the spectral theorem, it follows that $(x,y) = 0$; thus $M(N;G) \subset M(N;G')^\perp$. To show the other side of the inclusion, suppose by way of contradiction that there exists an $x \perp M(N;G)$ such that $\sigma(N;x) \cap G \neq \varnothing$. We can find a closed set $\delta_1 \subset G$ such that $\sigma(N;x) \cap \delta_1 \neq \varnothing$ and $\sigma(N;x)$ is not contained in the closure of δ_1'. Let $x = x_1 + x_2$ be the decomposition of this x with respect to $\mathcal{K} = E(\delta_1)\mathcal{K} + E(\delta_1')\mathcal{K}$. Since $x_1 \in M(N;\delta_1) \subset M(N;G)$ and $x \in M(N;G)^{\perp}$, x_1 must be zero. But $x_1 = 0$ implies that $\sigma(N;x) = \sigma(N;x_2) \subset$ the closure of δ_1', which contradicts the choice of δ_1. Thus $x \in M(N;G)^{\perp}$ implies that $\sigma(N;x) \subset G'$. Since G' is closed, $M(N;G')$ is a closed subspace and the proof is complete.

REMARK. If G is an open set, the above result shows that $E(G)\mathcal{K} = \overline{M(N;G)}$.

The following propositions are either explicitly (and/or implicitly) contained in Clancey and Wadhwa [**3**], Frunza [**10**], Radjabalipour [**11**], and Stampfli [**12**] and are stated here for easy reference.

PROPOSITION 1.2. *Let S be a subnormal operator on a Hilbert space $\mathcal{H}$ and let N be the m.n.e. of S on a Hilbert space $\mathcal{K}$. Suppose S^* has the s.v.e.p. Then $\tilde{\sigma}(S^*;x) \subset \tilde{\sigma}(N^*;x) = \sigma(N;x) \subset \sigma(S;x)$ for all $x \in \mathcal{H}$, where $\tilde{\sigma}$ denotes the complex conjugate of the set σ.*

For a normal operator N, the orthogonal complement of the space $\overline{M(N;G)}$ is $M(N;G')$ which is the same as $M(N^*;\tilde{G}')$. The following result will help to describe the situation for a subnormal decomposable operator.

PROPOSITION 1.3. *Let S be an operator on a Hilbert space $\mathcal{H}$. Suppose both S and S^* have s.v.e.p. Then $M(S^*;\tilde{G}') \subset M^{\perp}(S;G)$.*

PROOF. Let $x \in M(S^*;\tilde{G}')$, i.e., $\tilde{\sigma}(S^*;x) \subset G'$. In order to prove the result, it is enough to show that if $y \in M(S;G)$, then $(x,y) = 0$. Recall that $y_S(\lambda)$, $\lambda \notin \sigma(S;y)$, is the analytic extension of the function $(S-\lambda)^{-1}y$. For $\lambda \notin [\sigma(S;y) \cup \tilde{\sigma}(S^*;x)]$,

$$\begin{aligned}(y_S(\lambda), x) &= (y_S(\lambda), (S^* - \bar{\lambda})x_{S^*}(\bar{\lambda})) \\ &= ((S-\lambda)y_S(\lambda), x_{S^*}(\bar{\lambda})) = (y, x_{S^*}(\bar{\lambda})).\end{aligned}$$

Thus $h(\lambda) = (y_S(\lambda), x)$ for $\lambda \notin \sigma(S;y)$ and $h(\lambda) = (y, x_{S^*}(\bar{\lambda}))$ for $\bar{\lambda} \notin \sigma(S^*;x)$ is a well-defined function. Also $h(\lambda)$ is analytic for all $\lambda \notin \sigma(S;y)$ and $\lambda \notin \tilde{\sigma}(S^*;y)$. Since $\tilde{\sigma}(S^*;x) \cap \sigma(S;y)$ is empty, $h(\lambda)$ is an entire function; $\|h(\lambda)\| \to 0$ as $\lambda \to \infty$; hence $h(\lambda) \equiv 0$ which implies that $(x,y) = 0$.

The following is a result of Frunza which plays a crucial role in Frunza's elegant theorem that S is decomposable iff S^* is decomposable.

PROPOSITION 1.4 (FRUNZA). *If S is a decomposable operator on a Hilbert space $\mathcal{H}$, then $M(S;G)^{\perp}$ is a maximal spectral space of S^* and $\sigma(S^*|M(S;G)^{\perp}) \subset \tilde{G}'$. Thus $M(S;G)^{\perp} \subset M(S^*;\tilde{G}')$.*

This result of Frunza combined with Proposition 1.3 shows that if S is a decomposable operator on a Hilbert space $\mathcal{H}$, then

$$\mathcal{H} = \overline{M(S;G)} \oplus M(S^*;\tilde{G}'). \tag{1.2}$$

If S is decomposable, then S^* is also decomposable, so we have

$$\mathcal{H} = \overline{M(S^*;\tilde{G})} \oplus M(S;G'). \tag{1.3}$$

Now we have the main theorem of this section:

THEOREM 1.4. *Let S be a subnormal decomposable operator on a Hilbert space $\mathcal{H}$. Let G be an open set in* $\mathbf{C}$. *Then* $\mathcal{H} = \overline{M(S;G)} \oplus M(S;G') \oplus M^*$, *where* $M^* = M(S^*;\tilde{G}) \ominus M(S;G')$ *is a subspace of* $M(S^*;\partial\tilde{G})$ *and is invariant under* S^* *and* $\sigma(S^*|M^*) \subset \partial\tilde{G}$.

PROOF. By Proposition 1.2, since S is subnormal $\tilde{\sigma}(S^*;x) \subset \sigma(S;x)$; thus $M(S;G') \subset M(S^*;\tilde{G}')$. Combine it with the decomposition (1.2), i.e., $\mathcal{H} = \overline{M(S;G)} \oplus M(S^*;\tilde{G}')$, to get $\mathcal{H} = \overline{M(S;G)} \oplus M(S;G') \oplus M^*$ where $M^* = M(S^*;\tilde{G}') \ominus M(S;G')$. Obviously $\overline{M(S;G)} \oplus M(S;G')$ is invariant under S, hence M^* is invariant under S^*. Also for $x \in M^*$, $x \in M(S^*;\tilde{G}')$; therefore $\tilde{\sigma}(S^*;x) \subset G'$; x is orthogonal to $M(S;G')$, hence by the decomposition (1.3) the $\tilde{\sigma}(S^*;x) \subset$ closure of G'. Thus $\sigma(S^*;x) \subset \partial\tilde{G}$ for all $x \in M^*$.

REMARK. The Bergman shift S on $L_a^2(\Delta)$ is an irreducible operator (i.e., it has no nontrivial reducing subspace) and hence its dual T is also irreducible (Conway [**6**, Corollary 1.8]). Since T is decomposable, for any open set G such that $\sigma(T) \cap G$ and $\sigma(T) \cap G'$ are nonempty, the space $\overline{M(T;G)}$ and $M(T;G')$ are both nontrivial. Thus it follows from Theorem 1.4 that the corresponding space M^* will be nonzero. In particular, there will exist a nonzero vector f such that $\tilde{\sigma}(T^*;f) \subset \partial G$ (see also Corollary 3.1 of Clancey and Wadhwa [**3**]).

2. The minimal normal extension. Let S acting on a Hilbert space $\mathcal{H}$ be a subnormal operator and let $N = \int \lambda\, dE(\lambda)$ acting on $\mathcal{K} \supset \mathcal{H}$ be the minimal normal extension (m.n.e.) of S. If S is a subnormal and decomposable operator and if G is any open set such that $\sigma(S) \cap G \neq \emptyset$, then the space $\overline{M(S;G)}$ is a nonzero space contained in $M(S;\overline{G})$. We shall show that the m.n.e. of $S|\overline{M(S;G)}$ is the restriction of S to $E(G)\mathcal{K}$. The situation appears to be more complex for the m.n.e. of $S|M(S;\overline{G})$. We shall show that the m.n.e. of $S|M(S;\overline{G})$ is $E(G)\mathcal{K} + \mathcal{M}$, where $\mathcal{M}$ is a reducing subspace of $E(\partial G)\mathcal{K}$.

For any set G of the complex plane, $M(S;G) = M(S;G \cap \sigma(S))$. So without loss of generality we shall assume that $G \subset \sigma(T)$.

PROPOSITION 2.1. *Let S acting on $\mathcal{H}$ be a subnormal decomposable operator and let $N = \int \lambda\, dE(\lambda)$ acting on $\mathcal{K}$ be its m.n.e. Then the m.n.e. space of $S|\overline{M(S;G)}$ is $E(G)\mathcal{K}$.*

PROOF. Let $\mathcal{K}_1 = V\{N^{*j}f \colon f \in \overline{M(S;G)},\ j = 0,1,\ldots\}$ be the m.n.e. space of $S|\overline{M(S;G)}$. By the remark following Proposition 1.1, $E(G)\mathcal{K} = \overline{M(N;G)}$; also $\sigma(N,N^*f) \subset \sigma(N;f) \subset \sigma(S;f)$ for all $f \in \mathcal{H}$, thus $\mathcal{K}_1 \subset E(G)\mathcal{K}$. Now suppose for the sake of contradiction that $E(G)\mathcal{K} \ominus \mathcal{K}_1 = \mathcal{R}_1$ is a nonzero space. $E(G)\mathcal{R}_1 = \mathcal{R}_1$; therefore, by the regularity of the measure E, there exists a compact set $\delta \subset G$ such that $E(\delta)\mathcal{R}_1 = \mathcal{R}$ is nonzero. Now choose an open set G_1 such that $\overline{G}_1 \cap \delta$ is empty and $G \cup G_1 \supset \sigma(S)$. (The set G_1 can be chosen as follows: let $s = \operatorname{dist}(\delta, G')$ and let $G_1 = \{z \in \mathbf{C} \colon \operatorname{dist}(z,\delta) > s/2\}$.) Since S is decomposable, there exist maximal spectral subspaces $\mathcal{M}$ and $\mathcal{M}_1$ such that $\mathcal{H} = \mathcal{M} + \mathcal{M}_1$, where $\sigma(S|\mathcal{M}) \subset G$ and $\sigma(S|\mathcal{M}_1) \subset G_1$, that is $\mathcal{M} \subset \overline{M(S,G)}$ and $\mathcal{M}_1 \subset \overline{M(S;G_1)}$. Now the space $\mathcal{K}_1 + E(\overline{G}_1)\mathcal{K}$ reduces N and contains $\overline{M(S;G)} + M(S;\overline{G}_1)$ and hence it contains

$\mathcal{H}$. However $\mathcal{R}$ is orthogonal to $\mathcal{K}_1 + E(\overline{G}_1)\mathcal{K}$. Thus, unless $\mathcal{R}$ is zero, we have a contradiction to the assumption that N is the m.n.e. of S.

PROPOSITION 2.2. *Let S and N be as in Proposition 2.1. The spectrum of the m.n.e. of $S|M(S;\overline{G})$ is equal to $\overline{G}$.*

PROOF. Let $\mathcal{K}_1 = V\{N^{*j}f \colon f \in M(S;\overline{G}),\ j = 0,1,2,\ldots\}$. Then $N_1 = N|\mathcal{K}_1$ is the m.n.e. of $S|M(S;\overline{G})$. As in the preceding proof, $\mathcal{K}_1 \subset E(\overline{G})\mathcal{K}$, since $\mathcal{K}_1$ is a reduced N; $\sigma(N_1) \subset \overline{G}$. Now suppose, by way of contradiction, that $\sigma(N_1)$ is a proper subset of $\overline{G}$. From elementary point set topological considerations, it follows that $\exists\lambda \in G$ such that $\lambda \notin \sigma(N_1)$, thus we can find a closed set δ containing a neighborhood of λ such that $\delta \cap \sigma(N_1) = \emptyset$. Since $\delta \subset G \subset \sigma(S) = \sigma(N)$, $E(\delta)\mathcal{K}$ is not a zero space and $E(\delta)\mathcal{K}$ is orthogonal to $\mathcal{K}_1$. The contradiction can now be gotten by following the latter part of the proof of Proposition 2.1.

PROPOSITION 2.3. *Let S and N be as in Proposition 2.1. Then the m.n.e. space of $S|M(S;\overline{G}) = E(G)\mathcal{K} \oplus \mathcal{M}$, where $\mathcal{M}$ is a subspace of $E(\partial G)\mathcal{K}$.*

PROOF. Let $\mathcal{K}_1 = V\{N^{*j}f \colon f \in M(S;\overline{G}),\ j = 0,1,2,\ldots\}$ be the m.n.e. of $S|M(S;\overline{G})$. It is clear from the proof of the previous two propositions that $E(G)\mathcal{K} \subset \mathcal{K}_1 \subset E(\overline{G})\mathcal{K}$, from which the result follows.

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REFERENCES

1. C. Apostol, C. Foiaş and D. Voiculescu, *Some results on quasi-triangular operators*. IV, Rev. Roumaine Math. Pures Appl. **18** (1973), 487–514.
2. K. F. Clancey, *Seminormal operators*, Lecture Notes in Math., vol. 742, Springer-Verlag, New York, 1979.
3. K. F. Clancey and B. L. Wadhwa, *Local spectra of seminormal operators*, Trans. Amer. Math. Soc. **280** (1983), 415–428.
4. I. Colojoara and C. Foiaş, *The theory of generalized spectral operators*, Gordon & Breach, New York, 1968.
5. J. B. Conway, *Subnormal operators*, Pitman, Boston, Mass., 1981.
6. ——, *The dual of a subnormal operator*, J. Operator Theory **5** (1981), 195–211.
7. N. Dunford and J. T. Schwartz, *Linear operators*. II: *Spectral theory*, Interscience, New York, 1963.
8. ——, *Linear operators*. III: *Spectral operators*, Interscience, New York, 1971.
9. C. Foiaş, *Spectral maximal spaces and decomposable operators in Banach spaces*, Arch. Math. **14** (1963), 341-349.
10. S. Frunza, *A duality theorem for decomposable operators*, Rev. Roumaine Math. Pures Appl. **16** (1971), 1055–1058.
11. M. Radjabalipour, *On subnormal operators*, Trans. Amer. Math. Soc. **211** (1975), 377–389.
12. J. G. Stampfli, *Analytic extensions and spectral localization*, J. Math. Mech. **19** (1966), 287–296.
13. E. L. Stout, *The theory of uniform algebras*, Bogden and Quigley, Tarrytown, N. Y., 1971.

DEPARTMENT OF MATHEMATICS, CLEVELAND STATE UNIVERSITY, CLEVELAND, OHIO 44115