A ONE-ONE SELECTION THEOREM

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Abstract. Let $X, Y$ be Polish spaces without isolated points and $B \subseteq X \times Y$ a Borel set such that \{x: $B_x$ is nonmeager\} is comeager in $X$ and \{y: $B^y$ is nonmeager\} is comeager in $Y$. There is a comeager Borel $E \subseteq X$, a comeager Borel $F \subseteq Y$ and a Borel isomorphism $f$ from $E$ onto $F$ such that graph of $f \subseteq B$.

1. Introduction. In [3] Mauldin proved that if $B \subseteq [0,1] \times [0,1]$ is a Borel set such that $\lambda(\{x: \lambda(B_x) > 0\}) = 1$ and $\lambda(\{y: \lambda(B^y) > 0\}) = 1$, where $\lambda$ is the Lebesgue measure, then there exist Borel sets $E$ and $F$ of full measure and a Borel isomorphism $f$ from $E$ onto $F$ such that the graph of $f \subseteq B$. Our main theorem is a category analogue of this. Throughout this paper, $X, Y$ are taken to be Polish spaces without isolated points.

2. The main result. Our main theorem reads

Let $B \subseteq X \times Y$ be a Borel set such that \{x: $B_x$ is nonmeager\} is comeager in $X$ and \{y: $B^y$ is nonmeager\} is comeager in $Y$. Then there is a comeager Borel $E \subseteq X$, a comeager Borel $F \subseteq Y$ and a Borel isomorphism $f$ from $E$ onto $F$ such that the graph of $f = \{(x, y): y = f(x)\} \subseteq B$.

Our proof is analogous to that in [3] where several subsidiary results are proved, leading to the main theorem.

Theorem 1. Let $X = Y = [0,1]$, $B \subseteq X \times Y$ be a Borel set such that \{x: $B_x$ is comeager\} is comeager in $X$. There is a comeager Borel $E \subseteq X$, a meager Borel $F \subseteq Y$ and a Borel isomorphism $f$ from $E$ onto $F$ with graph $f \subseteq B$.

Proof. Fix an open base $\{U_n: n = 1,2,\ldots\}$ for $Y$ consisting of nonempty intervals.

Note that $B$ is comeager in $X \times Y$. Hence there exist dense open sets $V_1 \supseteq V_2 \supseteq \cdots$ in $X \times Y$ with $\bigcap_n V_n \subseteq B$.

By induction on $n$, we define a sequence of Borel sets $\{H_n: n = 1,2,\ldots\}$ such that for all $n$

1. $H_{n+1} \subseteq H_n \subseteq V_n$.

2. There exist a sequence $\{B_{n_i}: i = 1,2,\ldots\}$ of pairwise disjoint nonmeager $G_δ$ sets in $X$ with $B_{n_i} \subseteq (k_i/2^n, (k_i + 1)/2^n)$ for some integer $k_i$ and $\bigcup_i B_{n_i}$ comeager.

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in \( X \); a sequence \([a_{ni}, b_{ni}]: i = 1, 2, \ldots\) of pairwise disjoint closed intervals of length \( > 0 \) and \( \leq \frac{1}{2^n} \) and a nonempty open \( W_n \subseteq U_n \) with \( W_n \cap \bigcup_i [a_{ni}, b_{ni}] = \emptyset \) such that \( H_n = \bigcup_i B_{ni} X(a_{ni}, b_{ni}) \).

(3) For all \( x \), \( H_{n+1} x \subseteq H_{nx} \).

Now \( \cap_n H_n \) is the graph of the required function \( f \).

Construction of \( H_n \). Suppose \( H_n \) has been defined and equals \( \bigcup_i B_{mi} X(a_{mi}, b_{mi}) \) as in condition (2). For each \( i \), we define \( H_{n+1}^i \) so that \( \bigcup_i H_{n+1}^i = H_{n+1} \).

Fix \( i \). Note that \( V_{m+1} \cap (B_{mi} \times (a_{mi}, b_{mi})) \) is comeager in \( B_{mi} X(a_{mi}, b_{mi}) \). Hence by the Kuratowski-Ulam theorem, \( \{ y \in (a_{mi}, b_{mi}): V_{m+1}^y \cap B_{mi} \text{ is comeager in } B_{mi}\} \) is comeager in \( (a_{mi}, b_{mi}) \). Pick \( y_1, y_2 \) from this set with \( y_1 < y_2 \). Let \( M > \max(\{1/(y_2 - y_1), 1/l, 2^m\}) \), where \( l = \max(\text{length } U_{m+1} \cap (a_{mi}, b_{mi}), 1) \) and

\[
A_n^i = \left\{ x \in B_{mi}: \left[ y_1 - \frac{1}{4n}, y_1 + \frac{1}{4n} \right] \cup \left[ y_2 - \frac{1}{4n}, y_2 + \frac{1}{4n} \right] \subseteq V_{m+1} x \cap (a_{mi}, b_{mi}) \right\}, \quad n \geq M.
\]

Then \( A_n^i \) is coanalytic and \( \bigcup_{n \geq M} A_n^i = V_{m+1}^y \cap V_{m+1} y_1 \cap B_{mi} \) is a comeager Borel set in \( B_{mi} \).

Find pairwise disjoint Borel sets \( B_n^i \subseteq A_n^i \), \( n \geq M \), with \( U_n B_n^i = U_n A_n^i \). Put \( C_n^i = B_n^i \cap (k_i/2^n, (2k_i + 1)/2^{n+1}) \), \( D_n^i = B_n^i \cap ((2k_i + 1)/2^{n+1}, (k_i + 1)/2^n) \). Note that by possibly ignoring a meager set, we can suppose \( C_n^i \) and \( D_n^i \) to be nonmeager \( G_\delta \) sets in \( X \). Put

\[
H_{n+1}^i = \bigcup_{n \geq M} \left( C_n^i X \left( y_1 - \frac{1}{4n}, y_1 + \frac{1}{4n + 1} \right) \cup D_n^i X \left( y_2 - \frac{1}{4n}, y_2 - \frac{1}{4n + 1} \right) \right).
\]

To construct \( H_1 \), use \( V_1 \cap X \times (0,1) \) as a comeager open set in \( X \times Y \) and proceed as above.

**Corollary.** The previous theorem is true even when \( X \) and \( Y \) are arbitrary Polish spaces without isolated points.

**Proof.** Since the irrationals are homeomorphic to a comeager \( G_\delta \) subset of \([0,1]\), the result is true if \( X = Y = \text{irrationals} \).

Now any Polish space without isolated points contains a comeager \( G_\delta \) set homeomorphic to irrationals. Thus the result is true for \( X \), \( Y \) such spaces.

**Theorem 2.** Let \( B \subseteq X \times Y \) be such that \( \{ x: B_x \text{ is nonmeager } \} \) is comeager. Then there is a comeager Borel \( E \subseteq X \), a meager Borel \( F \subseteq Y \) and a Borel isomorphism \( f \) on \( E \) onto \( F \) such that \( \text{graph } f \subseteq B \).

**Proof.** Let \( U_1, U_2, \ldots \) be a countable open base for \( Y \). Let \( B_n^* = \{ x: B_x \cap U_n \text{ is comeager in } U_n \} \) and \( A_n = B_n^* - \bigcup_{m < n} B_m^* \). \( A_n \) is Borel for all \( n \) and \( \bigcap_n A_n \) is comeager in \( X \).

By ignoring a meager set if necessary, we can suppose that each \( A_n \) is a nonmeager \( G_\delta \). By induction on \( n \), we define \( f_n \) on \( E_n \subseteq A_n \). We then define \( f(x) = f_n(x) \) for \( x \in E_n \).
Suppose \( f_k, \ k \leq m, \) has been defined and range \( f_k \subseteq a \) meager \( F_a \) set, say \( F_k \subseteq U_k \). Put

\[
B_{m+1} = A_{m+1} \times \left( U_{m+1} - \bigcup_{i=1}^{m} F_i \right) \cap B.
\]

\( B_{m+1} \) is a Borel subset of \( A_{m+1} \times (U_{m+1} - \bigcup_{i=1}^{m} F_i) \) and \( \{ x: B_{m+1} \text{ is comeager (in } U_{m+1} - \bigcup_{i=1}^{m} F_i \} \} = A_{m+1}. \) By applying the previous result, get a comeager \( G_{m+1} \) in \( A_{m+1} \) and a Borel isomorphism \( f_{m+1} \) on \( E_{m+1} \) into \( U_{m+1} - \bigcup_{i=1}^{m} F_i \) such that range \( f_{m+1} \) is meager.

If \( f(x) = f_n(x) \) for \( x \in E_n \), \( f \) is a Borel isomorphism on \( \bigcup_n E_n \) into \( \bigcup_n F_n \). Thus domain \( f \) is comeager and the range is meager.

PROOF OF THE MAIN THEOREM. Find Borel sets \( E_1 \subseteq X, F_1 \subseteq Y \) such that \( E_1 \) is comeager, \( F_1 \) is meager and there is a Borel isomorphism \( h \) from \( E_1 \) onto \( F_1 \) satisfying graph \( h \subseteq B \).

Find Borel sets \( G \subseteq Y, H \subseteq X \) such that \( G \) is comeager, \( H \) is meager and there is a Borel isomorphism \( g \) from \( G \) onto \( H \) satisfying \( \{(x, y): x = g(y)\} \subseteq B - X \times F_1 \).

Define \( f \) on \( E_1 \cup H \) by

\[
f(x) = \begin{cases} 
  g^{-1}(x) & \text{if } x \in H, \\
  h(x) & \text{if } x \in E_1 - H.
\end{cases}
\]

Putting \( E = E_1 \cup H, \ F = \text{range } f \), we get the result.

REMARKS. In [3] Mauldin raises some interesting questions of which the following are still open to our knowledge.

1. If \( B \subseteq [0, 1] \times [0, 1] \) is a Borel set with \( B_x, B_y \) of positive Lebesgue measure for all \( x \) and \( y \), is there a Borel isomorphism of \( [0, 1] \) onto \( [0, 1] \) whose graph is a subset of \( B \)?

2. Is the category analog of the above true?

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