PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 2, June 1986

A ONE-ONE SELECTION THEOREM

H. SARBADHIKARI

ABSTRACT. Let X, Y be Polish spaces without isolated points and $B \subseteq X \times Y$ a Borel set such that $\{x\colon B_x$ is nonmeager$\}$ is comeager in X and $\{y\colon B^y$ is nonmeager$\}$ is comeager in Y. There is a comeager Borel $E \subseteq X$, a comeager Borel $F \subseteq Y$ and a Borel isomorphism f from E onto F such that graph of $f \subseteq B$.

1. Introduction. In [3] Mauldin proved that if $B \subset [0,1] \times [0,1]$ is a Borel set such that $\lambda\{x\colon \lambda(B_x) > 0\} = 1$ and $\lambda\{y\colon \lambda(B^y) > 0\} = 1$, where λ is the Lebesgue measure, then there exist Borel sets E and F of full measure and a Borel isomorphism f from E onto F such that the graph of $f \subseteq B$. Our main theorem is a category analogue of this. Throughout this paper, X, Y are taken to be Polish spaces without isolated points.

2. The main result. Our main theorem reads

Let $B \subset X \times Y$ be a Borel set such that $\{x\colon B_x$ is nonmeager$\}$ is comeager in X and $\{y\colon B^y$ is nonmeager$\}$ is comeager in Y. Then there is a comeager Borel $E \subseteq X$, a comeager Borel $F \subseteq Y$ and a Borel isomorphism f from E onto F such that the graph of $f = \{(x, y)\colon y = f(x)\} \subseteq B$.

Our proof is analogous to that in [3] where several subsidiary results are proved, leading to the main theorem.

THEOREM 1. *Let $X = Y = [0,1]$, $B \subseteq X \times Y$ be a Borel set such that $\{x\colon B_x$ is comeager$\}$ is comeager in X. There is a comeager Borel $E \subseteq X$, a meager Borel $F \subseteq Y$ and a Borel isomorphism f from E onto F with graph $f \subseteq B$.*

PROOF. Fix an open base $\{U_n\colon n = 1, 2, \dots\}$ for Y consisting of nonempty intervals.

Note that B is comeager in $X \times Y$. Hence there exist dense open sets $V_1 \supseteq V_2 \supseteq \cdots$ in $X \times Y$ with $\bigcap_n V_n \subseteq B$.

By induction on n, we define a sequence of Borel sets $\{H_n\colon n = 1, 2, \dots\}$ such that for all n

(1) $H_{n+1} \subseteq H_n \subseteq V_n, \dots$.

(2) There exist a sequence $\{B_{ni}\colon i = 1, 2, \dots\}$ of pairwise disjoint nonmeager G_δ sets in X with $B_{ni} \subseteq (k_i/2^n, (k_i + 1)/2^n)$ for some integer k_i and $\bigcup_i B_{ni}$ comeager

Received by the editors November 6, 1984 and, in revised form, May 20, 1985.
1980 *Mathematics Subject Classification*. Primary 04A15; Secondary 03E15, 54C65.
Key words and phrases. Comeager set, Borel isomorphism, graph of a function.

in X; a sequence $\{[a_{ni}, b_{ni}]: i = 1, 2, \dots\}$ of pairwise disjoint closed intervals of length > 0 and $\leqslant 1/2^n$ and a nonempty open $W_n \subseteq U_n$ with $W_n \cap \bigcup_i [a_{ni}, b_{ni}] = \varnothing$ such that $H_n = \bigcup_i B_{ni} X(a_{ni}, b_{ni})$.

(3) For all x, $\overline{H_{n+1x}} \subseteq H_{nx}$.

Now $\bigcap_n H_n$ is the graph of the required function f.

Construction of H_n. Suppose H_m has been defined and equals $\bigcup_i B_{mi} X(a_{mi}, b_{mi})$ as in condition (2). For each i, we define H^i_{m+1} so that $\bigcup_i H^i_{m+1} = H_{m+1}$.

Fix i. Note that $V_{m+1} \cap (B_{mi} \times (a_{mi}, b_{mi}))$ is comeager in $B_{mi} X(a_{mi}, b_{mi})$. Hence by the Kuratowski-Ulam theorem, $\{y \in (a_{mi}, b_{mi}): V^y_{m+1} \cap B_{mi}$ is comeager in $B_{mi}\}$ is comeager in (a_{mi}, b_{mi}). Pick y_1, y_2 from this set with $y_1 < y_2$. Let $M >$ maximum$\{1/(y_2 - y_1), 1/l, 2^m\}$, where $l = \max(\text{length } U_{m+1} \cap (a_{mi}, b_{mi}), 1)$ and

$$A^i_n = \left\{x \in B_{mi}: \left[y_1 - \frac{1}{4n}, y_1 + \frac{1}{4n}\right] \cup \left[y_2 - \frac{1}{4n}, y_2 + \frac{1}{4n}\right] \subseteq V_{m+1x} \cap (a_{mi}, b_{mi})\right\}, \qquad n \geqslant M.$$

Then A^i_n is coanalytic and $\bigcup_{n \geqslant M} A^i_n = V^{y_1}_{m+1} \cap V^{y_2}_{m+1} \cap B_{mi}$ is a comeager Borel set in B_{mi}.

Find pairwise disjoint Borel sets $B^i_n \subset A^i_n$, $n \geqslant M$, with $\bigcup_n B^i_n = \bigcup_n A^i_n$. Put $C^i_n = B^i_n \cap (k_i/2^n, (2k_i + 1)/2^{n+1})$, $D^i_n = B^i_n \cap ((2k_i + 1)/2^{n+1}, (k_i + 1)/2^n)$. Note that by possibly ignoring a meager set, we can suppose C^i_n and D^i_n to be nonmeager G_δ sets in X. Put

$$H^i_{m+1} = \bigcup_{n \geqslant M} \left(C^i_n X\left(y_1 - \frac{1}{4n}, y_1 - \frac{1}{4n+1}\right) \cup D^i_n X\left(y_2 - \frac{1}{4n}, y_2 - \frac{1}{4n+1}\right)\right).$$

To construct H_1, use $V_1 \cap X \times (0, 1)$ as a comeager open set in $X \times Y$ and proceed as above.

COROLLARY. *The previous theorem is true even when X and Y are arbitrary Polish spaces without isolated points.*

PROOF. Since the irrationals are homeomorphic to a comeager G_δ subset of $[0, 1]$, the result is true if $X = Y =$ irrationals.

Now any Polish space without isolated points contains a comeager G_δ set homeomorphic to irrationals. Thus the result is true for X, Y such spaces.

THEOREM 2. *Let $B \subseteq X \times Y$ be such that $\{x: B_x$ is nonmeager$\}$ is comeager. Then there is a comeager Borel $E \subseteq X$, a meager Borel $F \subseteq Y$ and a Borel isomorphism f on E onto F such that graph $f \subseteq B$.*

PROOF. Let $U_1, U_2, \dots$ be a countable open base for Y. Let $B^*_n = \{x: B_x \cap U_n$ is comeager in $U_n\}$ and $A_n = B^*_n - \bigcup_{m<n} B^*_m$. A_n is Borel for all n and $\bigcup_n A_n$ is comeager in X.

By ignoring a meager set if necessary, we can suppose that each A_n is a nonmeager G_δ. By induction on n, we define f_n on $E_n \subseteq A_n$. We then define $f(x) = f_n(x)$ for $x \in E_n$.

Suppose f_k, $k \leqslant m$, has been defined and range $f_k \subseteq$ a meager F_σ set, say $F_k \subseteq U_k$. Put

$$B_{m+1} = A_{m+1} X\left(U_{m+1} - \bigcup_{i=1}^{m} F_i\right) \cap B.$$

B_{m+1} is a Borel subset of $A_{m+1} X(U_{m+1} - \bigcup_{i=1}^{m} F_i)$ and $\{x\colon B_{m+1x}$ is comeager (in $U_{m+1} - \bigcup_{i=1}^{m} F_i)\} = A_{m+1}$. By applying the previous result, get a comeager $G_\delta E_{m+1}$ in A_{m+1} and a Borel isomorphism f_{m+1} on E_{m+1} into $U_{m+1} - \bigcup_{i=1}^{m} F_i$ such that range f_{m+1} is meager.

If $f(x) = f_n(x)$ for $x \in E_n$, f is a Borel isomorphism on $\bigcup_n E_n$ into $\bigcup_n F_n$. Thus domain f is comeager and the range is meager.

PROOF OF THE MAIN THEOREM. Find Borel sets $E_1 \subseteq X$, $F_1 \subseteq Y$ such that E_1 is comeager, F_1 is meager and there is a Borel isomorphism h from E_1 onto F_1 satisfying graph $h \subseteq B$.

Find Borel sets $G \subseteq Y$, $H \subseteq X$ such that G is comeager, H is meager and there is a Borel isomorphism g from G onto H satisfying $\{(x, y)\colon x = g(y)\} \subseteq B - X \times F_1$.

Define f on $E_1 \cup H$ by

$$\begin{aligned} f(x) &= g^{-1}(x) \quad \text{if } x \in H, \\ &= h(x) \quad \text{if } x \in E_1 - H. \end{aligned}$$

Putting $E = E_1 \cup H$, $F =$ range f, we get the result.

REMARKS. In [3] Mauldin raises some interesting questions of which the following are still open to our knowledge.

1. If $B \subseteq [0,1] \times [0,1]$ is a Borel set with B_x, B^y of positive Lebesgue measure for all x and y, is there a Borel isomorphism of $[0,1]$ onto $[0,1]$ whose graph is a subset of B?

2. Is the category analog of the above true?

ACKNOWLEDGMENT. I am grateful to Dr. R. D. Mauldin for bringing this problem to my attention and to Dr. S. M. Srivastava for simplifying the proof.

REFERENCES

1. S. Graf and R. D. Mauldin, *Measurable one-to-one selections and transition kernels*, Amer. J. Math. (to appear).

2. K. Kuratowski, *Topology*, Vol. I, Academic Press, New York; PWN, Warsaw, 1966.

3. R. D. Mauldin, *One-to-one selections—marriage theorems*, Amer. J. Math. **104** (1982), 823–828.

INDIAN STATISTICAL INSTITUTE, DIVISION OF THEORETICAL STATISTICS AND MATHEMATICS, 203 BARRACKPORE TRUNK ROAD, CALCUTTA 700 035, INDIA