PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 2, June 1986

A METRIC FORM OF MICROTRANSITIVITY

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ABSTRACT. We prove that every homogeneous compact metrizable space X has a compatible metric for which X is Lipschitz homogeneous and for which the group $L(X)$ of Lipschitz homeomorphisms of X acts Lipschitz microtransitively on X.

1. Introduction. This paper gives a metric analogue to what is called the "microtransitivity" of the action of the homeomorphism group of a compact metrizable homogeneous space. Homogeneous topological spaces have been studied by several authors after their definition by W. Sierpinski in [**12**]. A step forward was made possible in 1965 when Effros [**2**] proved a theorem concerning "smoothness" of the actions of Polish transformation groups. After its discovery in 1975 by Ungar [**13**] it has been used in several papers on homogeneous spaces written after 1975. As the work of Hagopian [**3, 4**], Ungar [**13, 14**], Jones [**7**], Rogers [**11**], Lewis [**8, 9**], Phelps [**10**] and Ancel [**1**] shows, many old problems have been solved and some old proofs have been simplified by the use of Effros's theorem in the form of microtransitivity.

For each compact metrizable space X, the group $H(X)$ of homeomorphisms of X onto itself (equipped with the compact-open topology) is a Polish transformation group on X. It follows from Effros [**2**, Theorem 2.1], that if G is a Polish transformation group acting transitively on a Polish space X, then for each $x \in X$ the map $T_x: G \to X$, given by $T_x(g) = g(x)$, is open. This implies that if (X, d) is a homogeneous compact metric space, then $H(X)$ acts microtransitively on X: For each $\delta > 0$ there is an Effros number $\varepsilon(\delta) > 0$ such that if $x, y \in X$, $d(x, y) < \varepsilon(\delta)$, then $f(x) = y$ for an $f \in H(X)$ satisfying $d(f(z), z) < \delta$ for all $z \in X$ (i.e., f is a δ-homeomorphism).

A Lipschitz homeomorphism is a surjective map $f: X \to Y$ of a metric space (X, d) onto a metric space (Y, σ) for which there is an $L \geqslant 1$ with

$$L^{-1}d(x, y) \leqslant \sigma(fx, fy) \leqslant Ld(x, y) \quad \text{for all } x, y \in X.$$

The least such L is denoted by bilip(f). If bilip(f) $\leqslant \lambda$, then f is called a λ-Lipschitz homeomorphism. For each $\lambda \geqslant 1$, $L_\lambda(X, d)$ denotes the set of all λ-Lipschitz homeomorphisms of (X, d) and we put $L(X, d) = \bigcup\{L_\lambda(X, d) \colon \lambda \geqslant 1\}$. We call (X, d) Lipschitz homogeneous [**15**] provided that for each pair x, y of points of X there is an $f \in L(X, d)$ such that $f(x) = y$. It has been shown in [**6**]

Received by the editors June 12, 1984 and, in revised form, April 2, 1985.
1980 *Mathematics Subject Classification*. Primary 54H15, 54E35.
Key words and phrases. Homogenous, Lipschitz homeomorphism, homeomorphism group.

that for every homogeneous compact metrizable space X there is a compatible metric d such that (X, d) is Lipschitz homogeneous.

Let (X, d) be a metric space. We say that the group $L(X, d)$ acts *Lipschitz microtransitively* on (X, d) if for each $\delta > 0$ and all $x \in X$ there is an $\varepsilon > 0$ with the following property: If $y \in X$ and $d(x, y) < \varepsilon$, then there is a δ-homeomorphism $f \in L(X, d)$ such that $f(x) = y$ and $\log(\operatorname{bilip}(f)) < \delta$. In particular, if X is compact, then $L(X, d)$ acts Lipschitz microtransitively on (X, d) iff for each $\delta > 0$ there is an $\varepsilon > 0$ such that the δ-homeomorphisms in $L_{1+\delta}(X, d)$ are "ε-transitive" on (X, d).

Given a Lipschitz homogeneous space (X, d), define for each pair x, y of points of X the number

$$L_{xy} = \inf\{\operatorname{bilip}(f) \colon f \in L(X, d),\ f(x) = y\}.$$

Thus, if $L(X, d)$ acts Lipschitz microtransitively on (X, d), then the formula $\sigma(x, y) = d(x, y) + \log(L_{xy})$ defines a compatible "inner" metric on X. (The converse is true for compact X; see Proposition 3.3.) However, there *are* nice compact Lipschitz homogeneous spaces (X, d) for which $L(X, d)$ does not act Lipschitz microtransitively on (X, d). Equip the Hilbert cube $Q = [-1, 1]^N$ with the metric d given by

$$d(x, y) = \max\{2^{-k}|x_k - y_k| \colon k \in N\}.$$

It has been shown in [**5**] that (Q, d) is Lipschitz homogeneous. On the other hand, it follows from [**15**, Theorem 3.4] that if $f\colon (Q, d) \to (Q, d)$ is any Lipschitz homeomorphism mapping the corner point $(1, 1, 1, \ldots)$ to the point $(0, 0, 0, \ldots)$, then necessarily $\operatorname{bilip}(f) \geqslant \sqrt{2}$. Virtually the same proof shows that for any Lipschitz homeomorphism $f \in L(X, d)$ mapping a point $(0, 0, 0, \ldots, 0, 1, 1, 1, \ldots)$ to the point $(0, 0, 0, \ldots)$, one has $\operatorname{bilip}(f) \geqslant \sqrt{2}$. Thus, $L(Q, d)$ does not act Lipschitz microtransitively on (Q, d). (However, there is an $L < \infty$ such that $L_{xy} \leqslant L$ for all $x, y \in Q$; see [**5**].) In this paper we prove that every homogeneous compact metrizable space X has a compatible metric d under which the action of $L(X, d)$ is Lipschitz microtransitive.

2. The main result. Our result will follow as a corollary to a sequence of five consecutive lemmas. In the sequel we assume that (X, d) is a homogeneous compact metric space. We consider $H(X)$ with the complete metric $\tilde{d}$ given by

$$\tilde{d}(f, g) = \sup\{d(fx, gx) + d(f^{-1}x, g^{-1}x) \colon x \in X\}.$$

Let $f \in H(X)$ and let $\varepsilon > 0$. We define the modulus of uniform continuity of f with respect to ε by setting

$$\omega_\varepsilon(f) = \sup\{\delta > 0 \colon d(x, y) < \delta \text{ implies } d(fx, fy),\ d(f^{-1}x, f^{-1}y) < \varepsilon\}.$$

Note that ω_ε is defined because each element of $H(X)$ is uniformly continuous. Hence, we obtain a map $\omega_\varepsilon\colon H(X) \to R$, where R denotes the set of real numbers.

2.1. LEMMA. *For each $\varepsilon > 0$, the ε-modulus of continuity $\omega_\varepsilon\colon H(X) \to R$ is lower semicontinuous.*

PROOF. Let $f \in H(X)$ and suppose that $\omega_\varepsilon(f) > t$. We must find a $\delta > 0$ such that $\tilde{d}(f, g) < \delta$ implies $\omega_\varepsilon(g) > t$. Choose an $\alpha > 0$ for which $\omega_\varepsilon(f) > t + \alpha$. Since X is compact, there is a $\delta > 0$ such that $d(x, y) < \omega_\varepsilon(f) - \alpha$ implies $d(fx, fy)$, $d(f^{-1}x, f^{-1}y) < \varepsilon - 2\delta$. Indeed, suppose that this does not hold. Then for each $n \in N$ we can find $x_n, y_n \in X$ such that $d(x_n, y_n) < \omega_\varepsilon(f) - \alpha$ but either $d(fx_n, fy_n) \geqslant \varepsilon - 1/n$ or $d(f^{-1}x_n, f^{-1}y_n) \geqslant \varepsilon - 1/n$. By passing to a subsequence if necessary, we can assume that $d(fx_n, fy_n) \geqslant \varepsilon - 1/n$ for all n.

Since X is compact, $\langle x_n \rangle$ has a subsequence $\langle x_{n_k} \rangle$ converging to a point x. Similarly, $\langle y_{n_k} \rangle$ has a subsequence $\langle y_{n_{k(l)}} \rangle$ converging to a point y. We have $d(x, y) \leqslant \omega_\varepsilon(f) - \alpha < \omega_\varepsilon(f)$, but $d(fx, fy) \geqslant \varepsilon$, which contradicts the definition of ω_ε.

To finish the proof, note that $d(x, y) < \omega_\varepsilon(f) - \alpha$ implies

$$d(gx, gy) \leqslant d(fx, gx) + d(fx, fy) + d(fy, gy) < \varepsilon$$

whenever $\tilde{d}(f, g) < \delta$, and similarly for $d(g^{-1}x, g^{-1}y)$. Thus, $\tilde{d}(f, g) < \delta$ implies $\omega_\varepsilon(g) > \omega_\varepsilon(f) - \alpha > t$. □

Let $\delta > 0$. A subset F of $H(X)$ is called *δ-transitive* if $d(x, y) \leqslant \delta$ implies that there is an $f \in F$ mapping x to y. Now let $\varepsilon > 0$ and suppose that F is δ-transitive. We put $X_\delta = \{(x, y) \in X \times X\colon d(x, y) \leqslant \delta\}$ and define a map $\varphi_\varepsilon^F\colon X_\delta \to R$ by

$$\varphi_\varepsilon^F(x, y) = \sup\{\omega_\varepsilon(f)\colon f \in F,\ f(x) = y\}.$$

2.2. LEMMA. *Let $\delta > 0$ and let F be an open δ-transitive subset of $H(X)$. Then for each $\varepsilon > 0$ the map φ_ε^F is lower semicontinuous.*

PROOF. The claim follows from the lower semicontinuity of ω_ε and Effros's theorem. To be precise, let $(x, y) \in X_\delta$ and let $\varphi_\varepsilon^F(x, y) > t$. We must find a neighbourhood U of (x, y) in X_δ such that $\varphi_\varepsilon^F(z, w) > t$ for all $(z, w) \in U$. It follows from the definition of φ_ε^F that there is an $f \in F$ with $\omega_\varepsilon(f) > t$ and $f(x) = y$. By 2.1 ω_ε is lower semicontinous, and hence we can find a neighbourhood V of f in F such that $g \in V$ implies $\omega_\varepsilon(g) > t$. Since F is open, we can choose a neighbourhood W of id in $H(X)$ satisfying the condition $gfh^{-1} \in V$ for all $g, h \in W$. By Effros's theorem $U = (T_x[W] \times T_y[W]) \cap X_\delta$ is a neighbourhood of (x, y) in X_δ. Now for all $(z, w) \in U$ there exist $g, h \in W$ with $g(x) = z$ and $h(y) = w$. Then $(hfg^{-1})(z) = w$ and $\omega_\varepsilon(hfg^{-1}) > t$. Consequently, $\varphi_\varepsilon(z, w) > t$ for all $(z, w) \in U$. □

2.3. LEMMA. *For each $\delta > 0$ and each δ-transitive neighbourhood U of* id *in $H(X)$, there is a compact δ-transitive subset K of $H(X)$ such that* id $\in K \subset U$.

PROOF. We can assume that U is open. Since X_δ is compact and $\varphi_{2^{-1}}^U$ is lower semicontinous by 2.2, we have

$$r_1 = \tfrac{1}{2}\inf\{\varphi_{2^{-1}}^U(x, y)\colon (x, y) \in X_\delta\} > 0.$$

Let $U_1 = \{f \in U\colon \omega_{2^{-1}}(f) > r_1\}$. Since $\omega_{2^{-1}}$ is lower semicontinuous, U_1 is open, and certainly id $\in U_1$ because $\omega_\alpha(g) \leqslant \omega_\alpha(\mathrm{id})$ for all $\alpha > 0$ and $g \in H(X)$. It immediately follows from the definition of U_1 that U_1 is δ-transitive.

We proceed inductively and assume that U_n is defined. Let

$$r_{n+1} = \tfrac{1}{2}\inf\{\varphi_{2^{-n-1}}^{U_n}(x, y)\colon (x, y) \in X_\delta\} > 0,$$

and put $U_{n+1} = \{f \in U_n: \omega_{2^{-n-1}}(f) > r_{n+1}\}$. In this way we obtain a sequence $\langle U_n \rangle$ of open δ-transitive neighbourhoods of id in $H(X)$.

Given $(x, y) \in X_\delta$, there is, for each $n \in N$, an $f_n \in U_n$ with $f(x) = y$. By construction the sequence $\langle f_n \rangle$ is equicontinuous. (Indeed, let $\varepsilon > 0$. Choose an $m \in N$ for which $2^{-m} < \varepsilon$. Then $d(z,w) < r_m$ implies $d(f_k z, f_k w) < \varepsilon$ for all $k \geqslant m$. For each $i \in \{1, \ldots, m-1\}$ choose an $\alpha_i > 0$ such that $d(z,w) < \alpha_i$ implies $d(f_i z, f_i w) < \varepsilon$. Let $\alpha = \min\{r_m, \alpha_1, \ldots, \alpha_{m-1}\}$. Then α witnesses the equicontinuity of $\langle f_n \rangle$ with respect to ε.) By Ascoli's theorem the sequence $\langle f_n \rangle$ is relatively compact as a subset of $H(X)$, and thus there exist a subsequence $\langle f_{n_k} \rangle$ and an $f \in H(X)$ such that $f_{n_k} \to f$. Clearly $f(x) = y$. Therefore, if we let K consist of all limits of sequences $\langle f_n \rangle$, where $f \in V_n$, then K is a δ-transitive closed subset of $H(X)$. Since $\omega_\alpha(g) \leqslant \omega_\alpha(\mathrm{id})$ for all $\alpha > 0$ and $g \in H(X)$, we have id $\in \bigcap\{V_n: n \in N\} \subset K$. Finally, K is equicontinuous. In fact, let $\varepsilon > 0$, let $f \in K$ and let $\langle f_n \rangle$ be a sequence such that $f_n \in V_n$ and $f_n \to f$. Choose an $m \in N$ with $2^{-m} < \varepsilon/3$. There is an $n \geqslant m$ such that $\tilde{d}(f_n, f) < \varepsilon/3$. But $d(x, y) < r_m$ implies

$$d(fx, fy) \leqslant d(fx, f_n x) + d(f_n x, f_n y) + d(f_n y, fy) < \varepsilon,$$

and thus r_m witnesses the equicontinuity of K with respect to ε.

For each finite sequence $i_1, \ldots, i_n$ of elements of N, let

$$a(i_1, \ldots, i_n) = (1 + 2^{-i_1})^{-1} \cdots (1 + 2^{-i_n})^{-1}.$$

We assume that the metric d on X is bounded above by 1. For each $n > 0$, let p_n be a positive integer such that $(1 + 2^{-n})^{p_n} < 1/n$. The proof of the following lemma is due to the referee and shortens the original proof given by the author.

2.4. LEMMA. *There is a sequence $\langle K_n \rangle$ of compact symmetric subsets of $H(X)$ such that each K_i contains the identity map and*

(1) *K_i is δ_i-transitive for some positive δ_i,*

(2) *for each $\varepsilon > 0$, the union of all $K_{i_1} \cdots K_{i_n}$ such that $a(i_1, \ldots, i_n) \geqslant \varepsilon$ is equicontinuous.*

PROOF. For each $r > 0$, let $G(r) = \{f \in H(X): \tilde{d}(f, \mathrm{id}) \leqslant r\}$. By induction we construct sequences $\langle r_n \rangle$ and $\langle \lambda_n \rangle$ of positive real numbers and a sequence $\langle K_n \rangle$ of compact symmetric subsets of $H(X)$ with the following properties.

(a) $2 p_n r_n \leqslant \min\{p_{n-1} r_{n-1}, \lambda_{n-1}, 1/n\}$.

(b) id $\in K_n \subset G(r_n)$, and K_n is δ_n-transitive for some $\delta_n > 0$.

(c) If $f_i \in K_1 \cup \cdots \cup K_n$ for $1 \leqslant i \leqslant p_n$, $g_i \in G(\lambda_n)$ for $1 \leqslant i \leqslant p_n + 1$, and $h = g_1 \circ f_1 \circ \cdots \circ g_{p_n} \circ f_{p_n} \circ g_{p_n+1}$, then $d(x, y) \leqslant \lambda_n$ implies $d(h(x), h(y)) \leqslant 1/n$ for all $x, y \in X$.

To begin, set $r_1 = 1$ and use Lemma 2.3 to obtain K_1 satisfying (b). Set $\lambda_1 = 1$. Since d is bounded above by 1, condition (c) is satisfied.

Now let $n > 1$ and inductively assume that we already have defined r_i, K_i and λ_i for $1 \leqslant i \leqslant n-1$. First choose $r_n > 0$ to satisfy (a). Next use Lemma 2.3 to choose K_n that satisfies (b). We select λ_n as follows. The equicontinuity of $K_1 \cup \cdots \cup K_n$

allows us to choose a decreasing sequence $\mu_0, \mu_1, \ldots, \mu_{p_n}$ of positive real numbers with the following properties:

(d) $\mu_0 = 1/n$.

(e) For $1 \leqslant i \leqslant p_n$ and for each $f \in K_1 \cup \cdots \cup K_n$, $d(x, y) \leqslant \mu_1$ implies $d(f(x), f(y)) \leqslant \mu_{i-1}/3$ whenever $x, y \in X$.

Now set $\lambda_n = \mu_{p_n}/3$. It is easily seen that (c) is valid for this choice of λ_n. This completes the inductive step and shows that the sequences $\langle r_n \rangle$, $\langle K_n \rangle$ and $\langle \lambda_n \rangle$ can be constructed as desired. Notice that (a) implies $r_n \leqslant 1/n$; hence the claim that $K_n \subset G(1/n)$ follows from (b).

For each $n > 0$, set $L_n = \bigcup\{K_{i_1} \cdots K_{i_r}\colon a(i_1, \ldots, i_r) \geqslant 1/n\}$. We prove the second contention of 2.4 by first establishing the weaker result that for every $h \in L_n$, $d(x, y) \leqslant \lambda_n$ implies $d(h(x), h(y)) \leqslant 1/n$ for all $x, y \in X$.

Thus, let $h \in L_n$. Then $h \in K_{i_1} \cdots K_{i_r}$ where $a(i_1, \ldots, i_r) \geqslant 1/n$. Since $(1 + 2^{-i}) \leqslant (1 + 2^{-n}) \leqslant (1/n)^{1/p_n}$ for $1 \leqslant i \leqslant n$, there are at most p_n indices i_j for which $i_j \leqslant n$. Therefore h decomposes as $h = g_1 \circ f_1 \circ \cdots \circ g_{p_n} \circ f_{p_n} \circ g_{p_n+1}$, where $f_k \in K_1 \cup \cdots \cup K_n$ for $1 \leqslant k \leqslant p_n$ and each g_k is the composition of a set of elements from $K_{n+1} \cup K_{n+2} \cup \cdots$ for $1 \leqslant k \leqslant p_n + 1$. For each $i > n$ the condition $(1 + 2^{-i})^{p_i} < 1/i < 1/n$ implies that there are at most p_i numbers $(1 + 2^{-i_j})$ $(1 \leqslant j \leqslant r)$ for which $i_j = i$. Hence, for each $i > n$, each g_k has at most p_i factors from K_i. Also $K_i \subset G(r_i)$ for $i > n$. Since $\tilde{d}(g_k, \mathrm{id}) \leqslant$ the sum of the distance from the factors of g_k to id, we have

$$\tilde{d}(g_k, \mathrm{id}) < \sum_{i=n+1}^{\infty} p_i r_i \quad \text{for } 1 \leqslant k \leqslant p_n + 1.$$

Condition (a) implies that

$$\sum_{i=n+1}^{\infty} p_i r_i \leqslant 2 p_{n+1} r_{n+1} \leqslant \lambda_n.$$

Thus $1 \leqslant k \leqslant p_n + 1$ implies $g_k \in G(\lambda_n)$. Now our assertion follows from condition (c).

To conclude the proof, let $\varepsilon > 0$ and set $L = \bigcup\{K_{i_1} \cdots K_{i_r}\colon a(i_1, \ldots, i_r) \geqslant \varepsilon\}$. We must prove that L is equicontinuous. Let $\mu > 0$. Choose $n \geqslant 1$ so that $1/n < \min\{\varepsilon, \mu\}$. Then $L \subset L_n$. Let $h \in L_n$. The assertion proved above implies that if $x, y \in X$ and $d(x, y) \leqslant \lambda_n$, then $d(h(x), h(y)) \leqslant 1/n \leqslant \mu$. We conclude that L is equicontinuous.

2.5. Lemma. *Let $\langle K_n \rangle$ be as in 2.4. Then there is a compatible metric σ of X such that the elements of K_n are $(1 + 2^{-n})$-Lipschitz homeomorphisms in (X, σ) for all $n \in N$.*

Proof. For each finite sequence $i_1, \ldots, i_n$ of positive integers, we define a pseudometric $\sigma_{i_1 \cdots i_n}$ by the formula

$$\sigma_{i_1 \cdots i_n}(x, y) = \sup\{d(fx, fy)\colon f \in K_{i_1} \cdots K_{i_n}\}.$$

Since $K_{i_1} \cdots K_{i_n}$ is compact and thus equicontinuous, $\sigma_{i_1 \cdots i_n}$ is a continuous pseudometric on X. On the other hand, id $\in K_{i_1} \cdots K_{i_n}$, and, hence, in fact, $\sigma_{i_1 \cdots i_n}$ is a metric. Let

$$\sigma(x, y) = \sup\left\{a(i_1, \ldots, i_n)\sigma_{i_1 \cdots i_n}(x, y) \colon i_1, \ldots, i_n,\ n \in N\right\}.$$

Obviously σ is a metric of X, stronger than d. To show that σ is compatible with the topology of X, we must prove that for each $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $\sigma(x, y) < \varepsilon$. By 2.4 there is a $\delta > 0$ such that if $a(i_1, \ldots, i_n) \geqslant \varepsilon/\operatorname{diam}(X)$ and $f \in K_{i_1} \cdots K_{i_n}$, then $d(x, y) < \delta$ implies $d(fx, fy) < \varepsilon$. Thus, $d(x, y) < \delta$ implies $a(i_1, \ldots, i_n)\sigma_{i_1 \cdots i_n}(x, y) < \varepsilon$ for all $i_1, \ldots, i_n \in N$, as desired.

Let $f \in K_n$, let $x, y \in X$ and let $\alpha > 0$. Choose $i_1, \ldots, i_m \in N$ such that

$$\sigma(fx, fy) \leqslant a(i_1, \ldots, i_m)\sigma_{i_1 \cdots i_m}(fx, fy) + \alpha.$$

We have

$$\begin{aligned}\sigma_{i_1 \cdots i_m}(fx, fy) &= \sup\left\{d(gfx, gfy) \colon g \in K_{i_1} \cdots K_{i_m}\right\} \\ &\leqslant \sup\left\{d(hx, hy) \colon h \in K_{i_1} \cdots K_{i_m} K_n\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\sigma(fx, fy) &\leqslant \frac{a(i_1, \ldots, i_m)}{a(i_1, \ldots, i_m, n)}\left(a(i_1, \ldots, i_m, n)\sigma_{i_1 \cdots i_m, n}(x, y)\right) + \alpha \\ &\leqslant (1 + 2^{-n})\sigma(x, y) + \alpha.\end{aligned}$$

Since K_n is symmetric (i.e, $K_n = K_n^{-1}$; this follows from the fact that $\omega_\alpha(g) = \omega_\alpha(g^{-1})$ for all $g \in H(X)$), we infer that each element of K_n is a $(1 + 2^{-n})$-Lipschitz homoemorphism of (X, σ). □

2.6. THEOREM. *For each homogeneous compact metrizable space X there is a compatible metric σ such that $L(X, \sigma)$ acts Lipschitz microtransitively on (X, σ).*

PROOF. Let $\langle K_n \rangle$ be the sequence promised by 2.4. Note that the proof of 2.4 shows that we can assume $K_n \subset G_n$, where $\langle G_n \rangle$ is a neighbourhood base of id in $H(X)$. Let σ be the metric given in 2.5. For every $\varepsilon > 0$ we can find an n such that the elements of K_n are ε-homeomorphisms of (X, σ) and $2^{-n} < \varepsilon$. Since X is compact and σ is compatible with the topology of X, K_n is δ-transitive for some $\delta > 0$. Thus, $\sigma(x, y) < \delta$ implies that there is an ε-homemorphism $f \in K_n$ such that $f(x) = y$ and $\operatorname{bilip}(f) \leqslant 1 + 2^{-n} < 1 + \varepsilon$. □

3. Topological microtransitivity. The reader might wonder whether the first half of Lipschitz microtransitivity, the topological microtransitivity of the group of Lipschitz homeomorphisms, is true for all compact Lipschitz homogeneous spaces. The answer is positive even though Effros's theorem cannot be applied, since, in general, $L(X, d)$ is *not* Polish for a compact (Lipschitz homogeneous) metric space (X, d).

3.1. THEOREM. *Let (X, d) be a Lipschitz homogeneous compact metric space. Then for each $x \in X$, the map $T_x \colon L(X, d) \to X$ is open.*

PROOF. Let U be a nonempty open subset of $L(X,d)$. To show that $T_x[U]$ is open, let $y \in T_x[U]$ with $f(x) = y$, $f \in U$. Choose a symmetric neighbourhood V of id such that $(\overline{V}^2)f \subset U$. Since $L(X,d)$ is separable and metrizable, there exists a sequence $\langle f_n \rangle$ of elements of $L(X,d)$ such that $L(X,d) = \bigcup\{f_nV\colon n \in N\}$. Since (X,d) is Lipschitz homogeneous, we have $T_x[L(X,d)] = X$ and thus

$$X = \bigcup\left\{T_y[f_nV]\colon n \in N\right\} = \bigcup\left\{f_n\left[T_y[V]\right]\colon n \in N\right\}.$$

Now X is of the second category in itself, and hence one of the sets $f_n[T_y[V]]$ is of the second category in X. Since the maps f_n are homeomorphisms, it follows that $T_y[V]$ is of the second category in X. Now $\overline{V} = \bigcup\{\overline{V} \cap L_n(X,d)\colon n \in N\}$. By Ascoli's theorem and the compactness of X each $L_n(X,d)$ is a compact subset of $L(X,d)$, and hence $T_y[\overline{V}]$ is σ-compact. There is m such that $T_y[\overline{V} \cap L_m(X,d)]$ is a (closed) second category subset of X, and thus $\operatorname{int}_X T_y[\overline{V}] \neq \varnothing$. Choose a $g \in \overline{V}$ for which $g(y) \in \operatorname{int}_X T_y[\overline{V}]$. Then $W = g^{-1}[\operatorname{int}_X T_y[\overline{V}]]$ is an open neighbourhood of y and $y \in W \subset T_y[\overline{V}^2] = T_x[\overline{V}^2 f] \subset T_x[U]$, which shows that $T_x[U]$ is open. □

Given an $x \in X$, denote by $L_x(X,d)$ the stabilizer of x in $L(X,d)$; i.e.,

$$L_x(X,d) = \{f \in L(X,d)\colon f(x) = x\}.$$

The natural map $L(X,d) \to L(X,d)/L_x(X,d)$ is open, and if (X,d) is compact and Lipschitz homogeneous, then by 3.1 the map $F_x\colon L(X,d)/L_x(X,d) \to X$ given by $F_x(gL_x(X,d)) = g(x)$ is open. Therefore, we obtain the following corollary (see [**13**, Theorem 3.2]).

3.2. COROLLARY. *Let (X,d) be a Lipschitz homogeneous compact metric space and let $x \in X$. Then X is a quotient of $L(X,d)$ by $L_x(X,d)$:*

$$L(X,d)/L_x(X,d) \cong X.$$

Now suppose that (X,d) is a compact metric space and that for every $r > 0$ there is a $\delta_r > 0$ such that $L_{1+r}(X,d)$ acts δ_r-transitively on (X,d). As in the proof of 3.1, one can show that for each neighbourhood U of id in $L(X,d)$ and for every $x \in X$, $T_x[L_{1+r}(X,d) \cap U]$ is a neighbourhood of x in X. This yields a proof of the following

3.3. PROPOSITION. *Let (X,d) be a compact metric space. Then $L(X,d)$ acts Lipschitz microtransitively on (X,d) iff the formula $\sigma(x,y) = d(x,y) + \log L_{xy}$ defines a compatible metric of X.*

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