CONCERNING THE FUNCTION EQUATION \( f(g) = f \), REGULAR MAPPINGS AND PERIODIC MAPPINGS

SAM W. YOUNG

ABSTRACT. Under certain conditions on the space \( X \), given \( f: X \to Y \) is light and \( g: X \to X \), the equation \( f(g) = f \) yields only periodic solutions for \( g \).

1. Introduction. In [M, p. 238], Mioduszewski proved the following theorem: If each of \( f \) and \( g \) is a mapping of \([0,1]\) onto itself, \( f \) is light and \( fg = f \), then \( g^2 = \text{id} \). Thus \( g \) is either the identity or an involution, the only types of periodic maps on \([0,1]\). In this paper we explore more general conditions under which the equation \( fg = f \) leads to the conclusion that \( g \) is periodic. These results are contained in Theorem 3.1. In §2 we establish some preliminary theorems. §4 contains related examples.

All spaces considered are compact metric and \( \rho \) denotes the sup metric on function spaces. A mapping \( f: X \to Y \) is called light iff each point inverse is totally disconnected. A mapping \( f: X \to X \) is called positively regular if the family of iterates \( \{f, f^2, f^3, \ldots\} \) is an equicontinuous family. If \( f \) is a homeomorphism, then \( f \) is regular iff the family of all iterates is equicontinuous. Also in case \( f \) is a homeomorphism, positively regular implies regular because of compactness. Lemma 2.2 will strengthen this. A mapping \( f: X \to X \) is periodic iff there exists a positive integer \( n \) such that \( f^n = \text{id} \) = the identity mapping on \( X \). The smallest such \( n \) is the period of \( f \). The double arrow denotes an onto mapping.

2. Preliminaries. In this section we establish the connection between the function equation \( fg = f \) and regular homeomorphisms.

LEMMA 2.1. If \( g \) is a positively regular mapping of a compact metric space \( X \) onto itself, then either \( g \) is a homeomorphism or there exists \( \delta > 0 \) such that \( \rho(g^i, \text{id}) \geq \delta \) for \( i = 1, 2, 3, \ldots \).

PROOF. If \( g \) is not a homeomorphism then there exists \( c \in X \) such that \( g^{-1}(c) \) is nondegenerate. Since the family \( \{g, g^2, g^3, \ldots\} \) is equicontinuous, no subsequence of \( \text{diam}(g^{-1}(c)), \text{diam}(g^{-2}(c)), \ldots \) converges to 0. It follows that there exists \( \delta > 0 \) and points \( a_i, b_i \in X \) such that \( g^i(a_i) = g^i(b_i) = c \) and \( d(a_i, b_i) \geq 2\delta, \) \( i = 1, 2, 3, \ldots \). Then for each \( i \),

\[
\rho(g^i, \text{id}) \geq \max\{d(g^i(a_i), \text{id}(a_i)), d(g^i(b_i), \text{id}(b_i))\} \\
= \max\{d(c, a_i), d(c, b_i)\} \geq \delta.
\]

It should be pointed out here that on compact metric spaces, a homeomorphism is regular iff it is almost periodic [GH].

Received by the editors April 26, 1985 and, in revised form, June 20, 1985.

1980 Mathematics Subject Classification. Primary 54C10, 54F62; Secondary 54F20, 57S10.

Key words and phrases. Function equation, regular mapping, periodic mapping, light mapping, finite graph, orientable surface.

©1986 American Mathematical Society

0002-9939/86 $1.00 + $.25 per page

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
LEMMA 2.2. If $g$ is a positively regular mapping of a compact metric space $X$ onto itself, then $g$ is a regular homeomorphism.

PROOF. We can regard the space of all mappings of $X$ onto $X$ as a topological semigroup using the composition operation and the metric $\rho$. By the Ascoli theorem, $\Gamma(g) = \text{cl}\{g, g^2, g^3, \ldots\}$ is a compact subset and so in fact is a compact subsemigroup. By a theorem of Numakura [N, Lemma 3, p. 102], $\Gamma(g) \supset K(g) = \text{the set of all cluster points of the sequence } \{g, g^2, g^3, \ldots\}$ is a (an abelian) group. The unit $e$ of $K(g)$ is an idempotent, $e^2 = e$. Since $e$ is a mapping of $X$ onto $X$, it follows that $e = \text{id}$. And since $e$ is the limit of a subsequence of $\{g, g^2, g^3, \ldots\}$ it follows from Lemma 2.1 that $g$ is a homeomorphism.

LEMMA 2.3. If each of $X$ and $Y$ is a locally connected compact metric space and $f: X \to Y$ is a light mapping, then the family of mappings $\{g: g: X \to X and \text{fg} = f\}$ is equicontinuous.

PROOF. Suppose to the contrary. Then there exists $\varepsilon_0 > 0, a_i, b_i \in X$ and $g_i$ such that $fg_i = f$, $d(a_i, b_i) < 1/\iota$ and $d(g_i(a_i), g_i(b_i)) \geq \varepsilon_0$, $i = 1, 2, 3, \ldots$. Wlog assume that there exists $P \in X$ such that $a_i \to P$ and $b_i \to P$. Since $X$ is locally connected at $P$, it follows that there exists a sequence of continua $A_1 \supset A_2 \supset A_3 \supset \cdots \supset \{P\}$ closing down on $P$ such that, for each $i$, $\text{diam}(g_i(A_i)) \geq \varepsilon_0$. Again wlog assume that there is a nondegenerate continuum $K \subset X$ such that $g_i(A_i) \to K$.

Now we have $fg_i(A_i) \to f(K)$ but since $fg_i(A_i) = f(A_i) \to f(P)$ it follows that $f(K) = f(P)$. This is not possible since $f$ is light.

THEOREM 2.4. If each $X$ and $Y$ is a locally connected compact metric space, $f: X \to Y$ is a light mapping and $g: X \to X$ is a mapping such that $fg = f$, then $g$ is a regular homeomorphism such that for each $x \in X$ the closure of the $g$-orbit of $x$ is totally disconnected.

PROOF. Suppose that $X, Y, f$ and $g$ are given and satisfy the hypothesis. Since for each $i = 1, 2, 3, \ldots, fg^i = f$, it follows from Lemma 2.3 and Lemma 2.2 that $g$ is a regular homeomorphism.

Suppose $x \in X$ and consider $O(x) = \{g^i(x): i = 0, \pm 1, \pm 2, \ldots\}$, the $g$-orbit of $x$. We have that, for each integer $i$, $fg^i(x) = f(x)$ and so $f(\text{cl}(O(x))) = f(x)$. Since $f$ is light, $\text{cl}(O(x)) \subset f^{-1}f(x)$ and must be totally disconnected.

3. The main theorem. The compact metric space $G$ is a finite graph iff $V = \{x: x \in G \text{ and order } (x) \neq 2\}$ is finite, $G \sim V$ has finitely many components and if $C$ is a component of $G \sim V$ then $\text{cl}(C)$ is either an arc or a simple closed curve.

Part (1) of Theorem 3.1 establishes a direct generalization of the theorem of Mioduszewski [M] referred to in the introduction. Parts (2) and (3) give similar results for some 2-dimensional spaces.

THEOREM 3.1. Suppose each of $X$ and $Y$ is a metric space, $f: X \to Y$ is a light mapping and $g: X \to X$ is a mapping such that $fg = f$. Then $g$ is a periodic homeomorphism if $X$ satisfies one of the following:

1. $X$ is a finite graph.
2. $X$ is a compact subset of the plane whose boundary is a finite graph.
3. $X$ is a compact connected orientable surface.
PROOF OF PART (1). By Theorem 2.4, $g$ is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. We will first consider the special case that $X$ is a simple closed curve. There are three possibilities [H, p. 129]: (a) If $g$ is order preserving and $X$ has a fixed point under $g$, then $g$ is the identity on $X$. (b) If $g$ is order preserving and $X$ has no periodic point under $g$, then $g^2 = \text{id}$. Case (b) is eliminated since the orbit of any point in $X$ would be dense in $X$, and case (c) gives $g^2 = \text{id}$. The remaining case is (a) with some point of period $n$ and here we have $g^n = \text{id}$.

Now for the general case let

$$V = \{x : x \in X \text{ and order } (x) \neq 2\}$$

and

$$\mathcal{C} = \{A : A \text{ is a component of } X \sim V\}.$$

Both $V$ and $\mathcal{C}$ are finite and the homeomorphism $g$ permutes the elements of $V$ and the elements of $\mathcal{C}$. There exists a positive integer $m$ such that $g^m$ takes each point of $V$ onto itself and each member of $\mathcal{C}$ onto itself. If $A \in \mathcal{C}$, there exists a positive integer $n(A)$ such that $g^{n(A)} = \text{id}$ on $\text{cl}(A)$. This is because the theorem of Mioduszewski [M] handles the case that $\text{cl}(A)$ is an arc and the argument given above handles the case that $\text{cl}(A)$ is a simple closed curve. Let $n$ be a common multiple of $m$ and $\{n(A) : A \in \mathcal{C}\}$ and then $g^n = \text{id}$ on $X$.

PROOF OF PART (2). Let us suppose that $\text{int}(X) \neq \emptyset$ since otherwise part (1) would apply. Under the hypothesis, $X$ would be locally connected and so by Theorem 2.4 $g$ is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. In [H, p. 127] it is shown that each component of $\text{int}(X)$ contains a simple closed curve which is invariant under some positive iterate of $g$. Let $K$ be a simple closed curve lying in a component $A$ of $\text{int}(X)$ and $m$ be a positive integer such that $g^m(K) = K$. With the mappings $f$ and $g^m$ restricted to $K$, $fg^m = f$ and so, by part (1), $g^m$ is periodic on $K$.

It is also shown in [H, p. 129] that if $K$ is a simple closed curve contained in $\text{int}(X)$ and if both $K$ and some point of $K$ are invariant under a positive iterate of $g$, then $g$ is periodic on the component of $X$ which contains $K$. Applying this theorem to $g^m$, we have that $g^m$ is periodic on $A$ and therefore $g$ is periodic on $A$. For each component $A$ of $\text{int}(X)$ let $n(A)$ be a positive integer such that $g$ restricted to $A$ is of period $n(A)$. Clearly also, $g$ is of period $n(A)$ on $\text{cl}(A)$.

Now $\text{bd}(X)$ is invariant under $g$ and, with the mappings $f$ and $g$ restricted to $\text{bd}(X)$, $fg = f$. By part (1) there is a positive integer $l$ such that $g$ is of period $l$ restricted to $\text{bd}(X)$. Let $n$ be a common multiple of $l$ and $\{n(A) : A \in \mathcal{C}\}$. Then $g^n = \text{id}$ on $X$.

PROOF OF PART (3). By Theorem 2.4, $g$ is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. The theorem of von Kerekjarto [K1] says that $g$ is periodic if $X$ is anything except a 2-sphere, torus, disc or annulus. If $X$ is a disc or annulus, then part (2) applies (or for a separate argument see [F and R]).

The regular homeomorphisms on the torus are characterized [K2] as being topologically equivalent to a product of standard rotations and such a homeomorphism
would either be periodic, or rotate an invariant simple closed curve \( K \) with a dense orbit in \( K \) or have a dense orbit. The latter cases cannot occur by Theorem 2.4.

The regular homeomorphisms on the 2-sphere are characterized [R] as being topologically equivalent to either a standard rotation or a reflection through the equator or a reflection through the equator followed by a standard rotation. The second type is periodic and each of the other types rotates an invariant simple closed curve [R]. In this case, as in the case of the torus, the rotation on the invariant simple closed curve must be a periodic rotation with some period \( n \). The period of \( g \) would be \( n \) in all cases except for a reflection followed by a rotation on the 2-sphere. In this case the period of \( g \) would be \( n \) if \( n \) is even and \( 2n \) if \( n \) is odd. This completes the proof.

The possibility of extending Theorem 3.1 to include cases of dimension greater than 2 seems difficult since very little is known about regular homeomorphisms in this setting. One approach to the study of regular homeomorphisms in dimension 3 can be found in [B].

4. Some examples. If \( X \) is a compact metric space and \( g: X \rightarrow X \) is periodic, then let \( Y \) be the space obtained by decomposing \( X \) into \( g \)-orbits. If \( f: X \rightarrow Y \) is the projection map then of course \( fg = f \) and, for any \( h: Y \rightarrow Y \), \( hfg = hf \). In a similar manner we construct the following

EXAMPLE 4.1. Let \( X \) be the locally connected space which is the union of a countably collection of arcs \( [a, b_1], [a, b_2], [a, b_3], \ldots \) which have only the point \( a \) in common and such that \( \text{diam}([a, b_i]) \rightarrow 0 \) as \( i \rightarrow \infty \). Let \( g: X \rightarrow X \) be the mapping such that \( g \) fixes \( a \) and permutes the first two arcs \( [a, b_1] \) and \( [a, b_2] \), permutes the next three arcs with period three, the next four with period four and so on.

Let \( Y \) be the space obtained by decomposing \( X \) into \( g \)-orbits. We see that \( Y \) is homeomorphic to \( X \). If \( f: X \rightarrow Y \) is the projection map, then \( fg = f \).

The mapping \( g \) is regular and pointwise periodic but not periodic. This example indicates that some sort of finite structure is necessary on \( X \) to conclude that \( g \) is periodic.

EXAMPLE 4.2. We use \( X, Y, f \) and \( g \) from Example 4.1. Note that the Hilbert Cube \( Q \cong X \times X \times X \times \cdots \). Let \( g^\infty: Q \rightarrow Q \) be the product \( g \times g \times g \times \cdots \) and let \( f^\infty: Q \rightarrow Q \) be \( f \times f \times f \times \cdots \). We have that \( fg = f \) and now \( g \) is regular but not pointwise periodic.

EXAMPLE 4.3. Let \( f: [0, 1] \rightarrow [0, 1] \) be any mapping which is not light. Let us say that \( [a, b] \) is a proper subinterval of \( [0, 1] \) such that \( f([a, b]) = c \in [0, 1] \). Then let \( g: [0, 1] \rightarrow [0, 1] \) be any mapping such that \( g: [a, b] \rightarrow [a, b] \) and \( g(x) = x \) for all \( x \not\in [a, b] \). Then \( fg = f \). This example indicates that the requirement that \( f \) be light is necessary to conclude that \( g \) is periodic.

REFERENCES


[H] E. Hemmingsen, Plane continua admitting non-periodic autohomeomorphisms with equi-


DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849