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SHORTER NOTES

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RANGES OF JOINT LAPLACE-FOURIER TRANSFORMS

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ABSTRACT. If $f \in L^1([0,\infty[\times\mathbf{R})$ and $\hat{f}(z,s) = \int_{-\infty}^{\infty}\int_0^{\infty} f(u,v)e^{iuz}e^{isv}\,du\,dv$, where $z \in H = \{z \in \mathbf{C}\colon \operatorname{Im} \geq 0\}$, $s \in \mathbf{R}$, then $\hat{f}(H\times\mathbf{R})\cup\{0\} = \hat{f}(\mathbf{R}\times\mathbf{R})\cup\{0\}$.

The proof is based on the following facts: (a) $\hat{f}$ is continuous on $H \times \mathbf{R}$, (b) for fixed $s \in \mathbf{R}$, $\hat{f}(z,s)$ is holomorphic on the interior of H, and (c) $\hat{f}$ vanishes at infinity on $H \times \mathbf{R}$. Since these facts are well known [**1**], I only check (c).

Define $e_x(u,v) = e^{-xu}$ for $x \in \mathbf{R}$ and $(u,v) \in [0,\infty[\times\mathbf{R}$. Fix $\varepsilon > 0$. By dominated convergence, there is an $R > 0$ such that $\|e_R f\|_1 < \varepsilon$. Then, for $\operatorname{Im} z \geq R$, we have

$$|\hat{f}(z,s)| \leq \int_{-\infty}^{\infty}\int_0^{\infty} |f(u,v)|e_{\operatorname{Im} z}(u,v)\,du\,dv \leq \|e_R f\|_1 < \varepsilon.$$

Again, by dominated convergence, the map $x \mapsto e_x f$ from $[0,R]$ into $L^1([0,\infty[\times\mathbf{R})$ is continuous. By compactness, there are $0 \leq x_1 < x_2 < \cdots < x_n \leq R$ such that, for any $0 \leq x \leq R$, there is an x_j with $\|e_x f - e_{x_j} f\|_1 < \varepsilon/2$. Since $\widehat{e_{x_j} f}$ is the Fourier transform, $\widehat{e_{x_j} f}$ vanishes at infinity on $\mathbf{R}\times\mathbf{R}$. Thus there is a compact $K \times L \subseteq \mathbf{R}\times\mathbf{R}$ such that $(r,s) \notin K\times L$ implies $|\widehat{e_{x_j} f}(r,s)| < \varepsilon/2$ for all x_j. Then, if $0 \leq y = \operatorname{Im} z \leq R$ and $(\operatorname{Re} z, s) \notin K \times L$, we have

$$\begin{aligned}|\hat{f}(z,s)| = |\widehat{e_y f}(\operatorname{Re} z,s)| &\leq |\widehat{e_y f}(\operatorname{Re} z,s) - \widehat{e_{x_j} f}(\operatorname{Re} z,s)| + |\widehat{e_{x_j} f}(\operatorname{Re} z,s)| \\ &\leq \|e_y f - e_{x_j} f\|_1 + |\widehat{e_{x_j} f}(\operatorname{Re} z,s)| < \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

for appropriate x_j. Thus $(z,s) \notin (K \times [0,R]) \times L$ implies that $|\hat{f}(z,s)| < \varepsilon$.

Evidently, we have $\hat{f}(\mathbf{R}\times\mathbf{R}) \cup (0) \subseteq \hat{f}(H\times\mathbf{R}) \cup \{0\}$. To prove the reverse inclusion, fix $r \notin \hat{f}(\mathbf{R}\times\mathbf{R}) \cup \{0\}$. Since $\hat{f}$ vanishes at infinity on $H \times \mathbf{R}$, there is an $s_0 \in \mathbf{R}$ such that $r \notin \hat{f}(H \times \{s_0\})$. Now set $A = \{s \in \mathbf{R}\colon r \notin \hat{f}(H\times\{s\})\}$. Then A is open. In fact, if $s \in A$, $\hat{f}(H\times\{s\})$ is bounded away from r. By uniform continuity, there is an $\varepsilon > 0$ such that $|s-t| < \varepsilon$ implies $r \notin \hat{f}(H\times\{t\})$, i.e., $t \in A$.

The complement of A is also open since, if $s \notin A$, there is a $z_0 \in H$ with $\hat{f}(z_0,s) = r$. By assumption on r, $\operatorname{Im} z_0 > 0$. Since $z \mapsto \hat{f}(z,s)$ is nonconstant and

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holomorphic, there is a circle C contained in the interior of H such that $\hat{f}(z,s)$ is never r for z in C. Then $a = \inf\{|\hat{f}(z,s) - r| : z \in C\} > 0$. By uniform continuity of $\hat{f}$, there is an $\varepsilon > 0$ such that $|s-t| < \varepsilon$ implies $|\hat{f}(z,s) - \hat{f}(z,t)| < a$ for all z in C. By Rouché's Theorem, $\hat{f}(z,t) - r$ has as many roots inside C as does $\hat{f}(z,s) - r$, i.e., at least one. Then $|s-t| < \varepsilon$ implies $t \notin A$.

Since $\mathbf{R}$ is connected and $s_0 \in A$, A is equal to $\mathbf{R}$, and so r does not belong to $\hat{f}(H \times \mathbf{R}) \cup \{0\}$.

REMARK. The foregoing proof is valid with $[0,\infty[\times\mathbf{R}$ replacd by $[0,\infty[\times\mathbf{R}^n$, and $H \times \mathbf{R}$ by $H \times \mathbf{R}^n$, etc.

Rewording the theorem gives

COROLLARY. *For $f \in L^1([0,\infty[\times\mathbf{R})$, the spectrum of f as an element of the Banach algebra $L^1([0,\infty[\times\mathbf{R})$ is the same as the spectrum of f as an element of the Banach algebra $L^1(\mathbf{R} \times \mathbf{R})$.*

REMARK. Note that the maximal ideal space of $L^1([0,\infty[\times\mathbf{R})$ is (homeomorphic to) $H \times \mathbf{R}$ and that of $L^1(\mathbf{R} \times \mathbf{R})$ is $\mathbf{R} \times \mathbf{R}$ (see [**1**]). Thus, the above corollary provides a Banach algebra B and a closed subalgebra A such that

(i) the spectrum of any element of A as an element of A coincides with the spectrum as an element of B, while

(ii) not every multiplicative linear functional on A extends to a multiplicative linear functional on B.

For another example where the range of a Fourier type transform is attained on a small portion of the domain see [**2**].

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