SHORTER NOTES

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RANGES OF JOINT LAPLACE-FOURIER TRANSFORMS

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ABSTRACT. If \( f \in L^1([0,\infty) \times \mathbb{R}) \) and \( \hat{f}(z,s) = \int_{-\infty}^{\infty} \int_{0}^{\infty} f(u,v) e^{iu z} e^{isv} du \, dv, \)
where \( z \in H = \{ z \in \mathbb{C} : \text{Im} \, z \geq 0 \}, s \in \mathbb{R} \), then \( \hat{f}(H \times \mathbb{R}) \cup \{0\} = \hat{f}(\mathbb{R} \times \mathbb{R}) \cup \{0\} \).

The proof is based on the following facts: (a) \( \hat{f} \) is continuous on \( H \times \mathbb{R} \), (b) for fixed \( s \in \mathbb{R} \), \( \hat{f}(z,s) \) is holomorphic on the interior of \( H \), and (c) \( \hat{f} \) vanishes at infinity on \( H \times \mathbb{R} \). Since these facts are well known [1], I only check (c).

Define \( e_x(u,v) = e^{-x u} \) for \( x \in \mathbb{R} \) and \( (u,v) \in [0,\infty) \times \mathbb{R} \). Fix \( \varepsilon > 0 \). By dominated convergence, there is an \( R > 0 \) such that \( \|e_R f\|_1 < \varepsilon \). Then, for \( \text{Im} \, z > R \), we have
\[
\int_{-\infty}^{\infty} \int_{0}^{\infty} |f(u,v)| e^{iz(u,v)} du \, dv < \|e_R f\|_1 < \varepsilon.
\]
Again, by dominated convergence, the map \( x \mapsto e_x f \) from \([0,R]\) into \( L^1([0,\infty) \times \mathbb{R}) \) is continuous. By compactness, there are \( 0 \leq x_1 < x_2 < \cdots < x_n \leq R \) such that, for any \( 0 \leq x \leq R \), there is an \( x_j \) with \( \|e_{x_j} f - e_x f\|_1 < \varepsilon/2 \). Since \( e_{x_j} \hat{f} \) is the Fourier transform, \( e_{x_j} \hat{f} \) vanishes at infinity on \( \mathbb{R} \times \mathbb{R} \). Thus there is a compact \( K \times L \subseteq \mathbb{R} \times \mathbb{R} \) such that \( (r,s) \notin K \times L \) implies \( \|e_{x_j} \hat{f}(r,s)\| < \varepsilon/2 \) for all \( x_j \). Then, if \( 0 \leq y = \text{Im} \, z \leq R \) and \( (\text{Re} \, z, s) \notin K \times L \), we have
\[
|\hat{f}(z,s)| = |\hat{e_y} \hat{f}(\text{Re} \, z, s)| \leq |\hat{e_y} \hat{f}(\text{Re} \, z, s) - e_{x_j} \hat{f}(\text{Re} \, z, s)| + |e_{x_j} \hat{f}(\text{Re} \, z, s)| < \varepsilon/2 + \varepsilon = \varepsilon
\]
for appropriate \( x_j \). Thus \( (z,s) \notin (K \times [0,R]) \times L \) implies that \( |\hat{f}(z,s)| < \varepsilon \).

Evidently, we have \( \hat{f}(\mathbb{R} \times \mathbb{R}) \cup \{0\} \subseteq \hat{f}(H \times \mathbb{R}) \cup \{0\} \). To prove the reverse inclusion, fix \( r \notin \hat{f}(\mathbb{R} \times \mathbb{R}) \cup \{0\} \). Since \( \hat{f} \) vanishes at infinity on \( H \times \mathbb{R} \), there is an \( s_0 \in \mathbb{R} \) such that \( r \notin \hat{f}(H \times \{s_0\}) \). Now set \( A = \{s \in \mathbb{R} : r \notin \hat{f}(H \times \{s\})\} \).
Then \( A \) is open. In fact, if \( s \in A \), \( \hat{f}(H \times \{s\}) \) is bounded away from \( r \). By uniform continuity, there is an \( \varepsilon > 0 \) such that \( |s-t| < \varepsilon \) implies \( r \notin \hat{f}(H \times \{t\}) \), i.e., \( t \in A \).

The complement of \( A \) is also open since, if \( s \notin A \), there is a \( z_0 \in H \) with \( \hat{f}(z_0,s) = r \). By assumption on \( r \), \( \text{Im} \, z_0 > 0 \). Since \( z \mapsto \hat{f}(z,s) \) is nonconstant and

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holomorphic, there is a circle $C$ contained in the interior of $H$ such that $\hat{f}(z, s)$ is never $r$ for $z$ in $C$. Then $a = \inf\{|\hat{f}(z, s) - r|: z \in C\} > 0$. By uniform continuity of $\hat{f}$, there is an $\varepsilon > 0$ such that $|s - t| < \varepsilon$ implies $|\hat{f}(z, s) - \hat{f}(z, t)| < a$ for all $z$ in $C$. By Rouché’s Theorem, $\hat{f}(z, t) - r$ has as many roots inside $C$ as does $\hat{f}(z, s) - r$, i.e., at least one. Then $|s - t| < \varepsilon$ implies $t \notin A$.

Since $\mathbb{R}$ is connected and $s_0 \in A$, $A$ is equal to $\mathbb{R}$, and so $r$ does not belong to $\hat{f}(H \times \mathbb{R}) \cup \{0\}$.

**Remark.** The foregoing proof is valid with $[0, \infty] \times \mathbb{R}$ replaced by $[0, \infty] \times \mathbb{R}^n$, and $H \times \mathbb{R}$ by $H \times \mathbb{R}^n$, etc.

Rewording the theorem gives

**Corollary.** For $f \in L^1([0, \infty] \times \mathbb{R})$, the spectrum of $f$ as an element of the Banach algebra $L^1([0, \infty] \times \mathbb{R})$ is the same as the spectrum of $f$ as an element of the Banach algebra $L^1(\mathbb{R} \times \mathbb{R})$.

**Remark.** Note that the maximal ideal space of $L^1([0, \infty] \times \mathbb{R})$ is (homeomorphic to) $H \times \mathbb{R}$ and that of $L^1(\mathbb{R} \times \mathbb{R})$ is $\mathbb{R} \times \mathbb{R}$ (see [1]). Thus, the above corollary provides a Banach algebra $B$ and a closed subalgebra $A$ such that

(i) the spectrum of any element of $A$ as an element of $A$ coincides with the spectrum as an element of $B$, while

(ii) not every multiplicative linear functional on $A$ extends to a multiplicative linear functional on $B$.

For another example where the range of a Fourier type transform is attained on a small portion of the domain see [2].

**Bibliography**


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