PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 3, July 1986

A WIENER INVERSION-TYPE THEOREM

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ABSTRACT. Let $W(D) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \mid \|f\|_1 = \sum_{n=0}^{\infty} |a_n| < +\infty\}$, $f(z)$ a function in $W(D)$ for which $f(0) = 1$, and M_f the operator of multiplication by $f(z)$ on $W(D)$. It is shown that if k and m are integers for which $0 \leq m \leq k-1$ and X_k^m is the closed subspace of $W(D)$ spanned by $\{z^{nk+i} \mid n = 0, 1, \ldots; i = 0, 1, \ldots, m\}$, then *$M_f$ is bounded below on $X_k^m \Leftrightarrow f(z)$ does not have $k-m$ distinct zeros in any set of the form $\{w^i z_0 \mid 0 \leq i \leq k-1;\ |z_0| = 1\}$, where w is a primitive kth root of unity.*

1. Let $W(D)$ denote the Wiener disc algebra consisting of those analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $\|f\|_1 = \sum_{n=0}^{\infty} |a_n| < +\infty$. The classical Wiener inversion theorem states that if $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, where $\sum_{n=-\infty}^{\infty} |a_n| < +\infty$, then $1/f(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$ with $\sum_{n=-\infty}^{\infty} |b_n| < +\infty \Leftrightarrow f(\theta) \neq 0$ for all θ [**3**, p. 196]. A simple corollary is that if $f(z) \in W(D)$ then *multiplication by $f(z)$ on $W(D)$ is an operator M_f which is bounded below $\Leftrightarrow f(z) \neq 0$ for all $|z| = 1$.* Consequently, a sequence in $W(D)$ of the form $\{z^n f(z)\}_{n=0}^{\infty}$ is a basic sequence which is equivalent to the basis $\{z^n\}_{n=0}^{\infty}$ for $W(D) \Leftrightarrow f(z) \neq 0$ for all $|z| = 1$ [**1**].

In [**2**] a result of considerably greater generality was proved. In the context of multiplication operators this result says that *if $f(z) \in W(D)$ and k is an integer ≥ 1 then the operator M_f of multiplication by $f(z)$ is bounded below on the closed subspace $[z^{nk}]_{n=0}^{\infty}$ of $W(D) \Leftrightarrow f(z)$ has no set of k zeros of the form $\{w^i z_0\}_{i=0}^{k-1}$, where $|z_0| = 1$ and w is a primitive kth root of unity.*

The purpose of this paper is not only to prove a still more general inversion theorem of this type, but to unify the results of [**1** and **2**] mentioned above by fitting them into a context in which they occur as the natural extreme cases of a general theory relating invertibility of M_f to the number of "cyclic zeros" of $f(z)$ on the circle $|z| = 1$. As a corollary we derive a characterization of certain cyclically shifted sequences in $W(D)$ which are basic and equivalent to the basis $\{z^n\}_{n=0}^{\infty}$, thereby completing work begun in [**1** and **2**].

2. Throughout the paper $f(z)$ will denote a function in $W(D)$ for which $f(0) = 1$, k will be a fixed integer ≥ 1, and w a primitive kth root of unity.

If m is an integer for which $0 \leq m \leq k-1$ let X_k^m denote the closed subspace of $W(D)$ spanned by the sequence $\{z^{nk+i} \mid n = 0, 1, 2, \ldots;\ i = 0, 1, \ldots, m\}$. Then for any $k \geq 1$, $X_k^0 = [z^{nk}]_{n=0}^{\infty}$, $X_k^{k-1} = W(D)$, and $\{X_k^m\}_{m=0}^{k-1}$ is an increasing tower of subspaces of $W(D)$ between these two "endpoints". Viewed from this perspective the following theorem simultaneously generalizes and unifies the results of [**1** and

Received by the editors July 11, 1985.
1980 *Mathematics Subject Classification*. Primary 47B37, 47B38, 46B15; Secondary 47A05, 46E15.
Key words and phrases. Wiener disc algebra, multiplication operator, inversion theorem, basic sequence, shift basic sequence.

2] by describing the invertibility of M_f on the spaces X_k^m in terms of the number of zeros of $f(z)$ lying in sets of the form $\{w^k z_0\}_{i=0}^{k-1}$ on the unit circle. Of particular interest is the way in which this dependence is a monotonic function of m.

THEOREM. *The operator M_f is bounded below on $X_k^m \Leftrightarrow f(z)$ does not have $k-m$ distinct zeros in any set of the form $\{w^i z_0 | 0 \le i \le k-1, |z_0| = 1\}$.*

PROOF. Suppose M_f is *not* bounded below on X_k^m. Then there is a sequence $\{g_j(z)\}_{j=1}^{\infty}$ in X_k^m for which $\|g_j\|_1 = 1$ for all j, but for which $\|f(z)g_j(z)\|_1 \to 0$. By definition of the space X_k^m each $g_j(z)$ can be written in the form $g_j(z) = \sum_{i=0}^m z^i h_i^{(j)}(z)$, where $h_i^{(j)}(z) \in [z^{nk}]_{n=0}^{\infty}$ for all i and j, and for which $\sum_{i=0}^m \|h_i^{(j)}\|_1 = \|g_j\|_1 = 1$ for all j. Since $h_i^{(j)}(z) \in [z^{nk}]_{n=0}^{\infty}$, we have $h_i^{(j)}(w^p z) = h_i^{(j)}(z)$ for all integers p, while from the condition $\sum_{i=0}^m \|h_i^{(j)}\|_1 = 1$ we may assume (taking a subsequence of $\{g_j\}$ if necessary) that $\max_{0\le i\le m} \inf_j \|h_i^{(j)}\|_1 > 0$. Therefore, setting $f(z)\cdot g_j(z) = q_j(z)$ for all j and using the observations above together with the assumption that $q_j \to 0$ as $j \to \infty$ we have that

$$(f(w^l z))\left(\sum_{i=0}^m w^{li} z^i h_i^{(j)}(z)\right) = q_j(w^l z) \xrightarrow{j} 0 \quad \text{for all } l = 0, 1, 2, \ldots, m.$$

Hence

$$\left(\prod_{i=0}^m f(w^i z)\right) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^m \\ 1 & w^2 & \cdots & w^{2m} \\ 1 & w^m & \cdots & w^{m^2} \end{bmatrix} \begin{bmatrix} h_0^{(j)}(z) \\ z h_1^{(j)}(z) \\ z^m h_m^{(j)}(z) \end{bmatrix}$$

$$= \begin{bmatrix} \prod_{i\ne 0} f(w^i z)\cdot q_j(z) \\ \prod_{i\ne 1} f(w^i z)\cdot q_j(wz) \\ \prod_{i\ne m} f(w^i z)\cdot q_j(w^m z) \end{bmatrix} \xrightarrow{j} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now since $1, w, w^2, \ldots, w^m$ are distinct, the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^m \\ 1 & w^m & \cdots & w^{m^2} \end{bmatrix}$$

has a nonzero (Vandermonde) determinant, and hence is invertible. It follows that

$$\left(\prod_{i=0}^m f(w^i z)\right) \begin{bmatrix} h_0^{(j)}(z) \\ z h_1^{(j)}(z) \\ \vdots \\ z^m h_m^{(j)}(z) \end{bmatrix} \xrightarrow{j} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and hence that the sequence $\{(\prod_{i=0}^m f(w^i z))\cdot(h_l^{(j)}(z))\}_{j=1}^{\infty}$ converges to zero in $W(D)$ for each $l = 0, 1, \ldots, m$.

In exactly the same way, if σ is *any* subset of $\{0, 1, 2, \ldots, k-1\}$ with cardinality $m+1$ then since $\{w^i\}_{i\in\sigma}$ is again a set of distinct numbers, we have $(\prod_{i\in\sigma} f(w^i z))(h_l^{(j)}(z)) \xrightarrow{j} 0$ in $W(D)$ for each $l = 0, 1, \ldots, m$, where, for some l,

$\inf_j \|h_l^{(j)}\|_1 > 0$ according to our earlier remarks. Thus $\{\prod_{i\in\sigma} f(w^i z) | \operatorname{card}\sigma = m+1\}$ has a common zero z_0 on the unit circle. For, if $A = \{f(\theta) | f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{-in\theta}$, with $\|f\| = \sum_{n=-\infty}^{\infty} |a_n| < +\infty\}$, then we can identify $W(D)$ with the subspace of A consisting of all such $f(\theta)$ with $a_n = 0$ for $n < 0$. The proper maximal ideals in A are sets of the form $\{f(\theta) | f(\theta_0) = 0\}$ for some $\theta_0 \in [0, 2\pi]$ [**3**, p. 195], so if $\{\prod_{i\in\sigma} f(w^i z) | \operatorname{card}\sigma = m+1\}$ did not have a common zero on the unit circle it would follow that the corresponding maximal ideal in A generated by this set must be all of A. Consequently, there would exist functions $\{r_\sigma(\theta)\}$ in A for which $1 = \sum_\sigma r_\sigma(\theta)(\prod_{k\in\sigma} f(w^k e^{i\theta}))$, and hence for which

$$h_l^{(j)}(e^{i\theta}) = \sum_\sigma r_\sigma(\theta) \prod_{k\in\sigma} f(w^k e^{i\theta}) h_l^{(j)}(e^{i\theta})$$

for all l and j. But by the remarks above this is impossible since the latter sum converges to zero in A as $j \to \infty$, yet $\inf_j \|h_l^{(j)}\|_1 > 0$. Hence we conclude there must be a number z_0 with $|z_0| = 1$ for which $\prod_{i\in\sigma} f(w^i z_0) = 0$ for all subsets σ of $\{0, 1, \ldots, k-1\}$ of cardinality $m+1$. But then $f(w^i z_0) \neq 0$ for at most m values of i, where $0 \le i \le k-1$, so $f(w^i z_0) = 0$ for at least $k-m$ such values of i, and we see that $f(z)$ has $k-m$ zeros in the set $\{w^i z_0\}_{i=0}^{k-1}$.

On the other hand, suppose that $f(w^i z_0) = 0$ for some $|z_0| = 1$ and for all i in a subset σ_0 of $\{0, 1, \ldots, k-1\}$ of cardinality $k-m$. In particular, suppose $f(z) = \prod_{i\in\sigma_0}(1 - z/w^i z_0)$ (the general case reduces easily to this one, as we show later). Letting $\sigma_0' = \{0, 1, \ldots, k-1\} \setminus \sigma_0$ we have the cardinality of $\sigma_0' = m$, while

$$1/f(z) = \frac{\prod_{i\in\sigma_0'}(1 - z/w^i z_0)}{1 - z^k/z_0^k} = \left[\prod_{i\in\sigma_0'}\left(1 - \frac{z}{w^i z_0}\right)\right]\left[\sum_{n=0}^{\infty} \frac{z^{nk}}{z_0^{nk}}\right] = \sum_{j=0}^{\infty} b_j z^j,$$

where

(i) $|b_j| = |b_{nk+j}|$ for all j and all n (since $|z_0| = 1$), and

(ii) $b_j = 0$ for $j \notin \{nk + i | n = 0, 1, \ldots;\ i = 0, 1, \ldots, m\}$ (since the degree of $\prod_{i\in\sigma_0'}(1 - z/w^i z_0)$ is equal to m).

By (ii), for any $r \ge 0$ the polynomial $\sum_{j=0}^{r} b_j z^j \in X_k^m$, while from (i) it follows that for any $\varepsilon > 0$ there is a constant c and an integer r for which $\|\sum_{i=0}^{r} c b_j z^j\|_1 = 1$ and $\sup_{0\le j\le r} |c b_j| < \varepsilon$. Since $f(z)$ is a polynomial of degree $k-m$ and $1/f(z) = \sum_{j=0}^{\infty} b_j z^j$, for any r the function $f(z) \cdot \sum_{j=0}^{r} b_j z^j$ is equal to $1 + z^{r+1} q_r(z)$, where $q_r(z)$ is some polynomial of degree $k-m-1$ whose coefficients (for any r) are bounded by $L = \sup_j |b_j| \cdot \|f(z)\|_1 < +\infty$. Hence, by the above, for any $\varepsilon > 0$ we can pick $|c| < \varepsilon$ and some r so that $\|\sum_{j=0}^{r} c b_j z^j\|_1 = 1$, but for which $\sup_j |c|\,|b_j| < \varepsilon$, and hence for which

$$\begin{aligned}
\left\|(f(z))\left(\sum_{j=0}^{r} c b_j z^j\right)\right\|_1 &\le |c|\,\|1 + z^{r+1} q_r(z)\|_1 \\
&\le |c|\left[1 + (k-m)\sup_j |b_j|\,\|f(z)\|_1\right] \\
&\le |c| + (k-m)\cdot\varepsilon\cdot\|f(z)\|_1 < \varepsilon[1 + (k-m)\|f(z)\|_1].
\end{aligned}$$

That is, M_f is not bounded below on X_k^m.

For the general case, suppose $f(z)$ is an arbitrary function in $W(D)$ for which $f(w^i z_0) = 0$ for some $|z_0| = 1$ and for all $i \in \sigma_0$, a subset of $\{0, 1, \ldots, k-1\}$ with cardinality $k - m$. Since the polynomials are dense in $W(D)$ and the subspace $X = \{g(z) \in W(D) | \, g(w^i z_0) = 0, \, i \in \sigma_0\}$ is of finite codimension in $W(D)$, the polynomials in X are also dense in X. It follows that $f(z)$ can be approximated as closely as desired by a polynomial of the form $q(z) = p_{\sigma_0}(z) r(z)$, where $p_{\sigma_0}(z) = \prod_{i \in \sigma}(1 - z/w^i z_0)$ (and $r(z)$ is some other polynomial). Hence if M_f were bounded below on X_k^m it would follow that M_q would also be bounded below on X_k^m for some such $q(z)$, an impossibility since we just showed that $M_{p_{\sigma_0}}$ is not bounded below on X_k^m. Thus if $f(z)$ has $k-m$ zeros in the set $\{w^i z_0 | \, 0 \le i \le k-1, \, |z_0| = 1\}$ then M_f cannot be bounded below on X_k^m, and the proof is complete.

From this theorem we get the following result which incorporates those of [**1** and **2**] into a unified description of cyclically shifted basic sequences in $W(D)$.

COROLLARY. *If $f(z) \in W(D)$ with $f(0) = 1$, k is an integer, and m is an integer for which $0 \le m \le k-1$, then a sequence in $W(D)$ of the form $\{z^{kn+i} f(z) | \, n = 0, 1, \ldots; \; i = 0, 1, \ldots, m\}$ is a basic sequence equivalent to the basis $\{z^n\}_{n=0}^{\infty}$ for $W(D) \Leftrightarrow f(z)$ does not have $k-m$ distinct zeros in any set of the form $\{w^i z_0 | \, 0 \le i \le k-1, \, |z_0| = 1\}$.*

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