A GEOMETRICAL CHARACTERIZATION OF SINGLY GENERATED DOUGLAS ALGEBRAS

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ABSTRACT. If $B$ is a Douglas algebra with $B \supsetneq H^\infty + C$, then $B$ is singly generated if and only if $\text{ball}(B/H^\infty + C)$ has an extreme point.

Let $H^\infty$ denote the space of boundary functions of bounded analytic functions in the open unit disk $D$. Let $L^\infty$ be the space of bounded measurable functions on $\partial D$ with respect to the normalized Lebesgue measure. A uniformly closed subalgebra between $H^\infty$ and $L^\infty$ is called a Douglas algebra. It is well known that $H^\infty + C$ is a Douglas algebra, where $C$ is the space of continuous functions on $\partial D$. By Chang and Marshall’s theorem [2, 7], a Douglas algebra is generated by $H^\infty$ and complex conjugate of some inner functions. It is an interesting problem to give a characterization of a Douglas algebra which is generated by complex conjugate of a single inner function (see [7 and 3, p. 398]). Such a Douglas algebra will be called singly generated. Up to now, its characterization has not been known. We shall give a geometrical characterization of a singly generated Douglas algebra. For a subset $F$ of $L^\infty$, we denote by $[F]$ the uniformly closed subalgebra generated by $F$.

THEOREM. Let $B$ be a Douglas algebra with $H^\infty + C \subsetneq B$. Then the following assertions are equivalent.

(i) There is an inner function $I$ such that $B = [H^\infty, I]$.

(ii) There is an extreme point of $\text{ball}(B/H^\infty + C)$.

For a Banach space $Y$, we denote by $\text{ball}(Y)$ the closed unit ball of $Y$. A point $x$ in $\text{ball}(Y)$ is called extreme if $\|x + y\| \leq 1$ and $y \in \text{ball}(Y)$ imply $y = 0$. Extreme points of $\text{ball}(B/H^\infty + C)$, where $B$ is a Douglas algebra, are studied in [6, 9].

For a Douglas algebra $B$, we write $M(B)$ as the maximal ideal space of $B$. We put $X = M(L^\infty)$. For a point $x$ in $M(H^\infty + C)$, we denote by $\mu_x$ the representing measure on $X$ for $x$, and by $\text{supp } \mu_x$ the support set for $\mu_x$. For an inner function $I$, we denote by $N(I)$ the weak*-closure in $X$ of $\cup \{\text{supp } \mu_x : I \notin \text{supp } \mu_x \}$. We put $QC = (H^\infty + C) \cap (H^\infty + C)$. For some point $x$ in $X$, $\{y \in X; f(y) = f(x) \text{ for every } f \in QC\}$ is called a QC-level set.

To show our theorem, we need some lemmas.

LEMMA 1 [5, THEOREM 1]. If $I$ is an inner function, then

1. $Q \subseteq N(I)$ or $Q \cap N(I) = \emptyset$ for every QC-level set $Q$, and
2. $I|_Q \notin H^\infty|_Q$ for every QC-level set $Q$ with $Q \subset N(I)$.

LEMMA 2 [5, PROOF OF COROLLARY 4]. Let $I_1$ and $I_2$ be inner functions. Then $N(I_1) \subseteq N(I_2)$ if and only if $[H^\infty, I_1] \subset [H^\infty, I_2]$.
PROOF OF THE THEOREM. The fact that (i) \(\Rightarrow\) (ii) is already pointed out in \[9\]. By \[9,\ Lemma 1\], for an inner function \(I\) there is an interpolating Blaschke product \(b\) such that \([H^\infty, I] = [H^\infty, b]\), and \(b + H^\infty + C\) is an extreme point of ball\((H^\infty, \mathbb{I})/H^\infty + C\) by \[6,\ Theorem 5\].

To show the converse assertion, suppose that (i) is not true. Let \(f \in B\) with \(\|f + H^\infty + C\| = 1\). Since \(H^\infty + C\) has the best approximation property \[1\], we may assume \(\|f\| = 1\). We shall show that \(f + H^\infty + C\) is not an extreme point of ball\((B/H^\infty + C)\). By Chang and Marshall's theorem, we have \([H^\infty, f] = [H^\infty, \mathbb{I}; I\) is an inner function with \(I \in [H^\infty, f]\)]\]. Then there is an inner function \(I_0\) such that \(\|I_0 f + H^\infty + C\| < 1\) and \(I_0 \in [H^\infty, f]\). we put \(\alpha = 1 - \|I_0 f + H^\infty + C\|\), then \(\alpha > 0\). We take a function \(h\) with

\[
(1) \quad h \in H^\infty + C \quad \text{and} \quad \|I_0 f + h\| = \|I_0 f + H^\infty + C\|.
\]

Since \([H^\infty, I_0] \subset B\), by our starting assumption there is an inner function \(J\) such that

\[
[H^\infty, I_0] \subset [H^\infty, J] \subset B.
\]

By Lemma 2, we get \(N(I_0) \subset N(J)\). By Lemma 2(1), there is a QC-level set \(Q\) such that \(Q \subset N(J)\) and \(Q \cap N(I_0) = \emptyset\). Then there is a function \(q\) in QC satisfying

\[
(2) \quad 0 \leq q \leq 1 \quad \text{on} \quad X, \quad q = 1 \quad \text{on} \quad Q
\]

and

\[
(3) \quad q = 0 \quad \text{on} \quad \text{a neighborhood of} \quad N(I_0).
\]

To show our assertion, it is sufficient to prove that

\[
(4) \quad Jq \in B \quad \text{and} \quad Jq \notin H^\infty + C,
\]

and

\[
(5) \quad \|f + \alpha Jq + H^\infty + C\| \leq 1.
\]

(4) follows easily from (2) and Lemma 1(2). We shall prove that \(\|f + \alpha Jq + H^\infty + C\| \leq 1\); the other will be obtained by the same way. Let us take a measure \(\mu\) on \(X\) satisfying

\[
(6) \quad \|\mu\| = 1 \quad \text{and} \quad \mu \perp H^\infty + C,
\]

that is, \(\mu\) is an annihilating measure for \(H^\infty + C\) having the unit total variation, and

\[
(7) \quad \|f + \alpha Jq + H^\infty + C\| = \int_X (f + \alpha Jq) \, d\mu.
\]

We put

\[
(8) \quad E = \{x \in X; q(x) = 0\}.
\]

Since \(q \in QC\), \(E\) is a peak set for \(H^\infty + C\). By the Glicksberg peak set theorem \[3\] and (6), we have \(\mu|_E \perp H^\infty + C\), so that

\[
(9) \quad \mu|_{X \setminus E} \perp H^\infty + C.
\]
By (3), we can take \( q_0 \in QC \) with \( q_0 = 0 \) on \( N(I_0) \) and \( q_0 = 1 \) on \( X \setminus E \). Since \( q_0 I_0 \text{supp} \mu_x \in H^\infty \text{supp} \mu_x \) for every \( x \in M(H^\infty + C) \), we have \( q_0 I_0 \in H^\infty + C \) by \([8]\), so that \( q_0 I_0 h \in H^\infty + C \) by (1). By (9), we get

\[
\int_{X \setminus E} I_0 h \, d\mu = 0.
\]

Then

\[
\|f + \alpha I_0 q + H^\infty + C\| = \int_X (f + \alpha I_0 q) \, d\mu \quad \text{by (7)}
\]
\[
= \int_E (f + \alpha I_0 q) \, d\mu + \int_{X \setminus E} (f + \alpha I_0 q) \, d\mu
\]
\[
= \int_E f \, d\mu + \int_{X \setminus E} (f + \alpha I_0 q + I_0 h) \, d\mu \quad \text{by (8) and (10)}
\]
\[
\leq \|\mu|_E\| + \|\mu|_{X \setminus E}\| \{\|f + I_0 h\|_{X \setminus E} + \alpha\} \quad \text{by } \|f\| \leq 1
\]
\[
\leq \|\mu|_E\| + \|\mu|_{X \setminus E}\| \{\|I_0 f + h\| + \alpha\}
\]
\[
\leq 1 \quad \text{by (1) and (6)}.
\]

Thus we get (5) and complete the proof.

REMARK. In the proof of (ii) \( \Rightarrow \) (i), we actually prove that if \( f \in B \), \( \|I_0 f + H^\infty + C\| < 1 \) and \( [H^\infty, I_0] \subsetneq B \) for some inner function \( I_0 \) with \( I_0 \in [H^\infty, f] \), then \( f + H^\infty + C \) is not an extreme point of \( \text{ball}(B/H^\infty + C) \). Consequently, if \( f + H^\infty + C \) is an extreme point of \( \text{ball}(B/H^\infty + C) \), and if \( I_0 \) is an inner function such that \( I_0 \in [H^\infty, f] \) and \( \|I_0 f + H^\infty + C\| < 1 \), then \( B = [H^\infty, I_0] \subset [H^\infty, f] \subset B \). Thus we get: If \( f + H^\infty + C \) is an extreme point of \( \text{ball}(B/H^\infty + C) \), then \( B = [H^\infty, f] \).

REFERENCES

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