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LINEAR PERTURBATIONS OF A NONOSCILLATORY SECOND ORDER EQUATION

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ABSTRACT. It is shown that the equation $(r(t)x')' + g(t)x = 0$ has solutions which behave asymptotically like those of a nonoscillatory equation $(r(t)y')' + f(t)y = 0$, provided that a certain integral involving $f - g$ converges (perhaps conditionally) and satisfies a second condition which has to do with its order of convergence. The result improves upon a theorem of Hartman and Wintner.

We consider the differential equation

$$(r(t)x')' + g(t)x = 0 \tag{1}$$

as a perturbation of

$$(r(t)y')' + f(t)y = 0, \tag{2}$$

under the following standing assumption.

ASSUMPTION A. Let r and f be real-valued and continuous, with $r > 0$, on $[a, \infty)$. Suppose that (2) is nonoscillatory at infinity. Let g be continuous and possibly complex-valued on $[a, \infty)$.

It is known [**1**, p. 355] that since (2) is nonoscillatory at infinity, it has solutions y_1 and y_2 which are positive on $[b, \infty)$ for some $b \geq a$ and satisfy the following conditions:

$$r(y_1y_2' - y_1'y_2) = 1, \tag{3}$$

$$\lim_{t\to\infty} \frac{y_2(t)}{y_1(t)} = \infty. \tag{4}$$

It is convenient to define

$$\rho = y_2/y_1 \tag{5}$$

on $[b, \infty)$. From (3) and (4),

$$\rho' = 1/ry_1^2 \quad \text{and} \quad \lim_{t\to\infty} \rho(t) = \infty. \tag{6}$$

Our objective is to extend the following theorem of Hartman and Wintner [**1**, p. 379].

THEOREM 1 (HARTMAN-WINTNER). *Suppose that*

$$\int^{\infty} y_1y_2|f - g|\,dt < \infty,$$

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or, more generally, that

$$\int^{\infty} (f-g) y_1^2 \, dt$$

converges (perhaps conditionally), and

$$\int^{\infty} \rho' \Gamma \, dt < \infty \tag{7}$$

(see (6)), where

$$\Gamma(t) = \sup_{r \geq t} \left| \int_r^{\infty} (f-g) y_1^2 \, ds \right|.$$

Then (1) has solutions x_1 and x_2 such that $x_i = (1+o(1))y_i$ and

$$\frac{r x_i'}{x_i} = \frac{r y_i'}{y_i} + o\left(\frac{1}{y_1 y_2}\right)$$

as $t \to \infty$, for $i = 1, 2$.

The following is our result. After proving it we will show that it is stronger than Theorem 1.

THEOREM 2. *Suppose that*

$$\int^{\infty} (f-g) y_1 y_2 \, dt \tag{8}$$

converges (perhaps conditionally), and

$$\sup_{r \geq t} \left| \int_r^{\infty} (f-g) y_1 y_2 \, ds \right| \leq \phi(t), \qquad t \geq a, \tag{9}$$

where $\phi(t) \to 0$ monotonically as $t \to \infty$. Define

$$G(t) = \int_t^{\infty} (f-g) y_1^2 \, ds. \tag{10}$$

Now suppose that

$$\int^{\infty} \rho' |G| \phi \, dt < \infty \tag{11}$$

and

$$\overline{\lim_{t \to \infty}} (\phi(t))^{-1} \int_t^{\infty} \rho' |G| \phi \, ds = A < 1/3. \tag{12}$$

Then (1) has a solution x_1 such that

$$x_1 = (1 + O(\phi)) y_1 \tag{13}$$

and

$$(x_1/y_1)' = O(\phi \rho'/\rho) \tag{14}$$

as $t \to \infty$, and a solution x_2 such that

$$x_2 = (1 + O(\phi_m)) y_2 \tag{15}$$

and

$$(x_2/y_2)' = O(\phi_m \rho'/\rho) \tag{16}$$

as $t \to \infty$, *where*

$$\phi_m = \max\{\phi, \hat{\phi}\}, \tag{17}$$

with

$$\hat{\phi}(t) = \frac{1}{\rho(t)} \int_b^t \rho' \phi \, ds. \tag{18}$$

Notice that integration by parts as in the proof of Abel's test for convergence of improper integrals shows that if (8) converges, then G exists and satisfies the inequality

$$|G(t)| \leq 2\phi(t)/\rho(t). \tag{19}$$

We use the contraction mapping theorem [**1**, p. 404] to establish the existence of x_1. If x_1 satisfies the equation

$$x_1(t) = y_1(t) + \int_t^\infty [y_2(s)y_1(t) - y_1(s)y_2(t)](f(s) - g(s))x_1(s)\, ds \tag{20}$$

on $[t_0, \infty)$ for some $t_0 \geq b$, then x_1 satisfies (1) on $[t_0, \infty)$, and can therefore be continued as a solution of (1) over $[a, \infty)$. Although this suggests an obvious choice of a transformation whose fixed point would be a solution of (1), it is convenient to work instead with a transformation whose fixed point turns out to be the relative error $z_1 = (x_1 - y_1)/y_1$. Rewriting (20) in terms of z_1 motivates us to consider the transformation $\mathcal{T}$ defined by $\mathcal{T}z = Q + \mathcal{L}z$, where

$$Q(t) = \int_t^\infty [y_2(s) - y_1(s)\rho(t)](f(s) - g(s))y_1(s)\, ds \tag{21}$$

and

$$(\mathcal{L}z)(t) = \int_t^\infty [y_2(s) - y_1(s)\rho(t)](f(s) - g(s))y_1(s)z(s)\, ds. \tag{22}$$

We will show that $\mathcal{T}$ is a contraction mapping of a certain Banach space B. It will then be routine to verify that if z_1 is the fixed point of $\mathcal{T}$ in this space and

$$x_1 = y_1(1 + z_1), \tag{23}$$

then x_1 is a solution of (1) which satisfies (13) and (14).

We need the following lemma, which is an elementary extension of Abel's test. For a proof, see [**2**, Lemma 1].

LEMMA 1. *Suppose* $u \in C[t_0, \infty)$ *and*

$$\int^\infty y_1 y_2 (f - g) u\, ds$$

converges (perhaps conditionally). Then the function

$$(\mathcal{L}u)(t) = \int_t^\infty [y_2(s) - y_1(s)\rho(t)](f(s) - g(s))y_1(s)u(s)\, ds$$

is in $C'[t_0,\infty)$, *and it satisfies the inequalities*

$$|(\mathcal{L}u)(t)| \le \sigma(t) \quad \textit{and} \quad |(\mathcal{L}u)'(t)| \le 2\sigma(t)\rho'(t)/\rho(t), \qquad t \ge t_0,$$

where

$$\sigma(t) = \sup_{r\ge t}\left|\int_r^\infty y_1y_2(f-g)u\,ds\right|.$$

Since (8) is assumed to converge, Lemma 1 with $u=1$ implies that

$$|Q(t)| \le \phi(t) \tag{24}$$

and

$$|Q'(t)| \le 2\phi(t)\rho'(t)/\rho(t). \tag{25}$$

(See (9) and (21).) This motivates us to let $\mathcal{T}$ act on the Banach space

$$B = \{z \in C'[t_0,\infty) | z = O(\phi),\ z' = O(\phi\rho'/\rho),\ t\to\infty\},$$

with norm

$$\|z\| = \sup_{t\ge t_0}\max\left\{\frac{|z|}{\phi}, \frac{|z'|\rho}{2\phi\rho'}\right\}. \tag{26}$$

We will show that $\mathcal{L}$ as defined in (22) is a contraction on B if t_0 is sufficiently large. Since (24) and (25) imply that $Q\in B$, it will then follow that $\mathcal{T}$ is a contraction on B.

Now suppose that $z\in B$, and consider the integral

$$I(t;z) = \int_t^\infty y_1y_2(f-g)z\,ds. \tag{27}$$

We will show that this integral converges and satisfies the inequality

$$|I(t;z)| < 3\|z\|m(t)\phi(t), \tag{28}$$

where

$$m(t) = \phi(t) + (\phi(t))^{-1}\int_t^\infty \rho'|G|\phi\,ds. \tag{29}$$

Then Lemma 1 with $u=z$ will imply that $\mathcal{L}z\in B$ and

$$\|\mathcal{L}z\| \le 3\|z\|\sup_{t\ge t_0} m(t). \tag{30}$$

However, (12) and (29) imply that $\overline{\lim}_{t\to\infty} m(t) < 1/3$; hence, we see from (30) that $\mathcal{L}$ is a contraction if t_0 is sufficiently large. Therefore, the proof of existence of x_1 is reduced to establishing (28). This is accomplished by rewriting (27) and integrating by parts:

$$\begin{aligned} I(t;z) &= -\int_t^\infty G'\rho z\,ds \quad \text{(see (5) and (10))} \\ &= G(t)\rho(t)z(t) + \int_t^\infty G(\rho z)'\,ds. \end{aligned}$$

This integration is valid, since $|G\rho z| \leq 2\|z\|\phi^2$ and $|(\rho z)'| \leq 3\|z\|\rho'\phi$, because of (19) and (26), and the integral on the right converges absolutely, and is dominated by

$$3\|z\| \int_t^\infty \rho' |G| \phi \, ds.$$

This implies (28). Hence, $\mathcal{L}z_1 = z_1$ for some $z_1 \in B$. We omit the routine verification that x_1 as defined by (23) has the stated properties.

Now we must show that (1) has a solution x_2 which satisfies (15) and (16). To this end, choose $b_1 \geq b$ (recall that $y_1, y_2 > 0$ on $[b, \infty)$), so that x_1 has no zeros on $[b_1, \infty)$. This is possible, because of (13). We can write

$$y_2(t) = y_1(t) \left(c + \int_{b_1}^t \frac{ds}{ry_1^2} \right), \qquad t \geq b,$$

for a suitable constant c. Now define

$$x_2(t) = x_1(t) \left(c + \int_{b_1}^t \frac{ds}{rx_1^2} \right), \qquad t \geq b_1.$$

Then x_2 satisfies (1) and, after some manipulations which use (5) and (6), we see that

$$x_2/y_2 = (x_1/y_1)(1 + \psi), \tag{31}$$

where

$$\psi(t) = (\rho(t))^{-1} \int_{b_1}^t \rho' \left[\left(\frac{y_1}{x_1} \right)^2 - 1 \right] ds.$$

Because of (13) and definition (18),

$$\psi = O(\hat{\phi}). \tag{32}$$

Differentiating (31) and invoking (13), (14), (17), and (32) yields (16). This completes the proof of Theorem 2.

Straightforward manipulations using (5), (6), (13), (14), (15), and (16) show that

$$rx_1'/x_1 = ry_1'/y_1 + O(\phi/y_1y_2)$$

and

$$rx_2'/x_2 = ry_2'/y_2 + O(\phi_m/y_1y_2)$$

as $t \to \infty$; hence, the conclusions of Theorem 2 imply those of Theorem 1. We will show that the assumptions of Theorem 2 are weaker than those of Theorem 1. Since (7) implies (11) and (12) (with $A = 0$) for any nonincreasing ϕ, we have only to show that the assumptions of Theorem 1 imply that (8) converges. To this end, we integrate by parts to obtain

$$\int_{t_1}^{t_2} (f - g) y_1 y_2 \, dt = -G\rho|_{t_1}^{t_2} + \int_{t_1}^{t_2} \rho' G \, ds.$$

Therefore, because of (7), the convergence of (8) will be established if we show that $\lim_{t\to\infty} G(t)\rho(t) = 0$. If this were not so, there would be a $\gamma > 0$ and an increasing sequence $\{t_j\}$ of points in $[a, \infty)$ such that $\lim_{j\to\infty} t_j = \infty$ and

$$\Gamma(t_j)\rho(t_j) \geq \gamma, \quad \rho(t_{j-1}) < \rho(t_j)/2, \qquad j = 1, 2, \ldots.$$

Then

$$\int_{t_0}^{\infty} \rho'(s)\Gamma(s)\,ds = \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} \rho'(s)\Gamma(s)\,ds$$
$$\geq \sum_{j=1}^{\infty} \Gamma(t_j)[\rho(t_j) - \rho(t_{j-1})] > \sum_{j=1}^{\infty} \frac{\gamma}{2} = \infty,$$

which contradicts (7).

REMARK. If $\phi(t) \to 0$ sufficiently slowly by comparison with $1/\rho$ as $t \to \infty$, then $\phi_m = O(\phi)$. For example, this is true if $\phi\rho^\mu$ is eventually nondecreasing for some $\mu < 1$.

EXAMPLE. Consider the equation

$$(33) \qquad x'' + K[t^{-1}(\log t)^{-\alpha} \sin t]x = 0 \qquad (K, \alpha = \text{nonzero constants}),$$

as a perturbation of $y'' = 0$. Then $y_1 = 1$, $y_2 = t$, and

$$f(t) - g(t) = -Kt^{-1}(\log t)^{-\alpha} \sin t.$$

Integration by parts shows that if $\alpha > 0$ and $\beta \geq 0$, then

$$(34) \qquad \int_t^{\infty} s^{-\beta}(\log s)^{-\alpha} \sin s\,ds = t^{-\beta}(\log t)^{-\alpha}(\cos t + O(t^{-1})), \qquad t \to \infty.$$

Now suppose $\varepsilon > 0$. Then (34) implies that if a is sufficiently large, then (9) holds with

$$(35) \qquad \phi(t) = (K + \varepsilon)(\log t)^{-\alpha}$$

(take $\beta = 0$), while G in (10) satisfies the inequality

$$(36) \qquad |G(t)| \leq (K + \varepsilon)t^{-1}(\log t)^{-\alpha}, \qquad t \geq a$$

(take $\beta = 1$). Moreover, (34) with $\beta = 1$ also implies that

$$\overline{\lim_{t\to\infty}}\, t(\log t)^{\alpha}|G(t)| = K,$$

which precludes (7) if $\alpha < 2$; hence, Theorem 1 does not apply unless $\alpha \geq 2$, in which case it implies only that (33) has solutions x_1 and x_2 such that $x_1(t) = 1 + o(1)$, $x_1'(t) = o(t^{-1})$, $x_2(t) = t + o(t)$, and $x_2'(t) = 1 + o(1)$ as $t \to \infty$. On the other hand, (35) and (36) imply (12) with $A = K + \varepsilon$ if $\alpha = 1$, or $A = 0$ if $\alpha > 1$. Therefore, Theorem 2 implies that (1) has solutions x_1 and x_2 such that

$$x_1(t) = 1 + O((\log t)^{-\alpha}), \qquad x_1'(t) = O(t^{-1}(\log t)^{-\alpha}),$$

and

$$x_2(t) = t[1 + O((\log t)^{-\alpha})], \qquad x_2'(t) = [1 + O((\log t)^{-\alpha})]$$

as $t \to \infty$, provided that either $\alpha > 1$ (K arbitrary) or $\alpha = 1$ and $K < \frac{1}{3}$.

REFERENCES

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