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JOINTLY QUASINORMAL ISOMETRIES

MARY EMBRY-WARDROP AND RICHARD J. FLEMING

ABSTRACT. If U and V are isometries each of which commutes with U^*V and V^*U, then a necessary and sufficient condition that U and V commute is that the ranges of U and V are equal. This result leads to the construction of a subnormal-valued analytic function which has no normal extension.

In [**2**] Globevnik and Vidav proved that if f is an analytic function whose values are normal operators on a Hilbert space X, then the range of f is abelian. In [**1**] Fleming and Jamison ask if this result is valid when the values of a function are subnormal or even quasinormal. A related question is whether an analytic subnormal-valued function has an extension to an analytic normal-valued function. The answer to each of these questions is no, as will be seen in Example 2.

A sufficient condition that the values of an analytic function f be quasinormal is that $A(B^*C) = (B^*C)A$ whenever A, B and C are coefficients of f. If this condition holds and A and B are coefficients of f, then each of A and B commutes with each of A^*A, A^*B, B^*A and B^*B, in which case we shall call A and B *jointly quasinormal.* For the simple analytic function $f(z) = A + zB$ we can now paraphrase the question in [**1**] by asking whether A and B commute when A and B are jointly quasinormal. The answer to this question is also no, as will be seen in Example 1.

The key to the answers of the above-mentioned questions is in the following theorem concerning isometries. The terminology used in the paper is as follows: A is *normal* if A commutes with A^*, *quasinormal* if A commutes with A^*A, an *isometry* if $A^*A = I$, and a *partial isometry* if A^*A is a projection. Basic facts concerning these special operators can be found in [**3**]. The range of an operator A is denoted by $A(X)$.

THEOREM. *If U and V are jointly quasinormal isometries, the following are equivalent*:
(i) $UV = VU$,
(ii) $UV(X) = VU(X)$,
(iii) $U(X) = V(X)$.

PROOF. (i)$\Rightarrow$(ii) trivially. To see that (ii)$\Rightarrow$(iii) note that if $UV(X) = VU(X)$ then $U^*UV(X) = U^*VU(X)$. Consequently, $V(X) = U(U^*V)(X)$ since U is an isometry and U and V are jointly quasinormal. Thus, $V(X) \subset U(X)$ if $UV(X) = VU(X)$ and by symmetry $U(X) \subset V(X)$ also. To see that (iii)$\Rightarrow$(i) assume that $U(X) = V(X)$ or equivalently $UU^* = VV^*$ since U and V are (partial) isometries. Let $K = UV - VU$ and note that $K(X) \subset U(X)$ since $V(X) \subset U(X)$. Furthermore, $U^*K = U^*UV - U^*VU = V - UU^*V$ (since U is an isometry and U and V are

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jointly quasinormal) $= V - VV^*V$ (since $VV^* = UU^*$) $= 0$ (since V is an isometry). Therefore, $U^*K = 0$, so that $K(X)$ is orthogonal to $U(X)$. We previously showed $K(X) \subset U(X)$ also. These two results imply that $K = 0$ or that $UV = VU$, as desired. Q.E.D.

This theorem makes the task of constructing noncommuting jointly quasinormal operators easy. Alan Lambert first suggested the simple construction in Example 1.

EXAMPLE 1. Let X be a Hilbert space with orthonormal basis $\{e_n : n = 1, 2, \ldots\}$. Let U and V be the isometries for which $Ue_n = e_{2n}$ and $Ve_n = e_{2n-1}$. U and V do not commute since $UVe_1 = e_2$ and $VUe_1 = e_3$. On the other hand, $U^*V = V^*U = 0$ since $U(X)$ and $V(X)$ are orthogonal. Since $U^*U = V^*V = I$, all of the commutation properties for the joint quasinormality of U and V are satisfied trivially.

EXAMPLE 2. Let U and V be the noncommuting jointly quasinormal isometries in Example 1 and define $f(z) = U + zV$ for each complex number z. Note that $f(z)^*f(z) = (1 + |z|^2)I$ where I is the identity operator, so that each value $f(z)$ is quasinormal, and consequently, subnormal. Thus we have an example of a subnormal-valued analytic function with nonabelian range. To see that this also provides us with a subnormal-valued analytic function which does not have a normal-valued analytic extension we need only recall that such an extension would have an abelian range [**2**]. This, of course, would force f to have an abelian range.

We close with two observations. If A and B are jointly quasinormal operators with canonical polar decompositions UP and VQ, respectively, then U and V are jointly quasinormal partial isometries. (The proof of this depends upon a rather lengthy, but elementary, algebraic computation.) Moreover, the jointly quasinormal operators A and B commute exactly when U and V commute. Secondly, it follows easily from the Theorem that when U and V are jointly quasinormal partial isometries, a necessary and sufficient condition for U and V to commute is that $UV(X) = VU(X)$. Thus, the general question of commutativity of quasinormal operators reduces to consideration of their partially isometric factors.

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DEPARTMENT OF MATHEMATICS, CENTRAL MICHIGAN UNIVERSITY, MT. PLEASANT, MICHIGAN 48858