PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 3, July 1986

# CUBE SLICING IN $\mathbf{R}^n$

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ABSTRACT. We prove that every $(n-1)$-dimensional section of the unit cube in $\mathbf{R}^n$ has volume at most $\sqrt{2}$. This upper bound is clearly best possible.

**1. Introduction.** Hensley [3] showed that if $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbf{R}^n$ is the central unit cube in $\mathbf{R}^n$, and $H$ is an $(n-1)$-dimensional subspace of $\mathbf{R}^n$, then $|H \cap Q_n|$ lies between 1 and 5. (Here and henceforth $|\cdot|$ will denote the usual volume in a Euclidean space.) Hensley conjectured that the volume is at most $\sqrt{2}$. Besides proving this conjecture, we shall provide a somewhat simpler proof that the volume is at least 1.

The bound 5, obtained by Hensley, can be improved in a more general setting. In [1], we show that there is an absolute constant, $c$, so that for all $n$, every $n$-dimensional Banach space has a representation on $\mathbf{R}^n$, with unit ball, $B$, say, with the property that if $H$ and $K$ are $(n-1)$-dimensional subspaces of $\mathbf{R}^n$, $|H \cap B| \leqslant c|K \cap B|$. We can take $c$ to be $\sqrt{6}$.

Let $H$ be an $(n-1)$-dimensional subspace of $\mathbf{R}^n$ and $a = (a_1, \ldots, a_n) \in \mathbf{R}^n$ be the unit vector normal to $H$. Assume $n \geqslant 2$ and $|a_i| < 1$, for all $i$. Define $f = f_H$: $\mathbf{R} \to [0, \infty)$ by $f(r) = |(H + ra) \cap Q_n|$, so $f(r)$ is the volume of the intersection of $Q_n$ with a hyperplane parallel to $H$, at distance $|r|$ from $H$. The assumption that $n \geqslant 2$, $|a_i| < 1$ for all $i$ ensures that $H$ is not parallel to a face of $Q_n$ and hence that $f$ is continuous.

As in [3] we shall use probabilistic methods to obtain an expression for $|H \cap Q_n| = f(0)$. Let $X_1, \ldots, X_n$ be independent random variables, each uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$, with respect to a probability, $P$. Then the random vector $(X_1, \ldots, X_n)$ induces the usual measure on $Q_n$, and, since $f$ is continuous,

$$\begin{aligned} f(r) &= \lim_{s \to 0} \frac{1}{2s} \int_{r-s}^{r+s} f(t)\, dt \\ &= \lim_{s \to 0} \frac{1}{2s} \left| \left\{ x \in Q_n : \left| \sum_1^n a_i x_i - r \right| \leqslant s \right\} \right| \\ &= \lim_{s \to 0} \frac{1}{2s} P\left( \left| \sum_1^n a_i X_i - r \right| \leqslant s \right), \end{aligned}$$

Received by the editors July 10, 1985.
1980 *Mathematics Subject Classification*. Primary 52A20, 52A40, 60E05.

so $f$ is the continuous probability density function of the random variable $X = \sum a_i X_i$.

It was observed by Hensley that $f$ attains its maximum at 0. This is also an immediate consequence of the Brünn-Minkowski inequality, namely,

$$\left|\tfrac{1}{2}A + \tfrac{1}{2}B\right| \geqslant \left(\tfrac{1}{2}|A|^{1/k} + \tfrac{1}{2}|B|^{1/k}\right)^k$$

for convex sets $A$ and $B$ in $\mathbf{R}^k$. For, taking $k = n - 1$,

$$A = H \cap (Q_n + ra) \subset H = \mathbf{R}^k, \quad \text{and} \quad B = H \cap (Q_n - ra) = -A,$$

the convexity of $Q_n$ gives

$$H \cap Q_n \supset \tfrac{1}{2}(A + B) = \tfrac{1}{2}(A - A);$$

so using the Brünn-Minkowski inequality gives

$$f(0) = |H \cap Q_n| \geqslant \left|\tfrac{1}{2}(A - A)\right| \geqslant |A| = f(r).$$

The upper bound, $\sqrt{2}$, will thus apply to all sections of $Q_n$ by hyperplanes, not merely those by subspaces.

Our principal aim is to prove best possible bounds on $\|f\|_\infty = f(0) = |H \cap Q_n|$, with $(a_i)_1^n$, $(X_i)_1^n$ and $f$ as above.

**2. The lower bound.** The lower bound will be an immediate consequence of the following simple probabilistic inequality, which compares a random variable with a uniformly distributed random variable of the same $L_p$-norm.

LEMMA 1. *Let $Y$ be a random variable with a probability density function $g$: $\mathbf{R} \to [0, \infty)$. Then for $p > 0$,*

$$\|g\|_\infty \|Y\|_p \geqslant \tfrac{1}{2}(p + 1)^{-1/p}.$$

*Equality holds in the above if and only if $Y$ is uniformly distributed on some subinterval $[-t, t]$ of $\mathbf{R}$.*

PROOF. Without loss of generality, we may assume that $X$ is symmetric. Then setting

$$G(x) = \int_0^x g(t)\,dt \quad \text{for } x \geqslant 0,$$

we have $G(0) = 0$, $G(\infty) = 1/2$, and $G(x) \leqslant x\|g\|_\infty$, so

$$\begin{aligned} 2^{-p} = 2(G(\infty))^{p+1} &= 2\int_0^\infty \left(G(x)^{p+1}\right)' dx \\ &= 2(p+1)\int_0^\infty g(x)G(x)^p\,dx \\ &\leqslant (p+1)\|g\|_\infty^p \cdot 2\int_0^\infty g(x)\cdot x^p\,dx \\ &= (p+1)\|g\|_\infty^p \|Y\|_p^p. \end{aligned}$$

There is equality if and only if $g(t) = \|g\|_\infty$ or 0 almost everywhere.

THEOREM 2. *Every section of the unit cube, $Q_n$, in $\mathbf{R}^n$ by an $(n-1)$-dimensional subspace, $H$, has volume at least* 1. *The volume is* 1 *if and only if $H$ is parallel to a face of the cube.*

PROOF. Plainly it suffices to show that, with $(a_i)_1^n$, $(X_i)_1^n$ as above ($|a_i| < 1$ for all $i$) and $f$ the continuous probability density function of $X = \sum a_i X_i$, we have $|H \cap Q_n| = f(0) > 1$. Since the $X_i$'s are independent and identically distributed,

$$EX^2 = \left(\sum a_i^2\right) EX_1^2 = EX_1^2 = 1/12.$$

So $\|X\|_2 = 1/2\sqrt{3}$, and the result follows by applying Lemma 1 with $p = 2$ and $g = f$.

We should remark that Theorem 2 was conjectured by A. Good, as was the following extension to subspaces of arbitrary dimension. Let $U$ be a subspace of $\mathbf{R}^n$: then $|U \cap Q_n| \geqslant 1$. This extension was proved by Vaaler [**4**].

**3. The upper bound.** The proof of the principal result depends on the following lemma, whose proof we postpone until after the theorem itself.

LEMMA 3.

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \left|\frac{\sin t}{t}\right|^p dt \leqslant \frac{\sqrt{2}}{\sqrt{p}} \quad \text{if } p \geqslant 2,$$

*and there is equality if and only if $p = 2$.*

THEOREM 4. *Every section of the unit cube, $Q_n$, by an $(n-1)$-dimensional subspace, $H$, has volume at most $\sqrt{2}$. This upper bound is attained if and only if $H$ contains an $(n-2)$-dimensional face of $Q_n$.*

PROOF. Let $(a_i)_1^n$ be as above and assume $a_i \geqslant 0$ for all $i$. Suppose first that for some $i$, $a_i > 1/\sqrt{2}$; say $a_1 > 1/\sqrt{2}$. Consider $H \cap C_n$, where $C_n$ is the "cylinder" $\{x \in \mathbf{R}^n\colon |x_i| \leqslant 1/2, 2 \leqslant i \leqslant n\}$. Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection onto the subspace $\{x \in \mathbf{R}^n\colon x_1 = 0\}$, and $S$ the orthogonal projection onto $H$. Then $T(H \cap C_n)$ is the unit cube in $\mathbf{R}^{n-1}$ and

$$|H \cap C_n| = |T(H \cap C_n)| \frac{\|Se\|}{\|TSe\|} = \frac{\|Se\|}{\|TSe\|},$$

where $e$ is the unit vector $(1, 0, \ldots, 0)$ and $\|\cdot\|$ is the usual Eucidean norm in $\mathbf{R}^n$.

Since $Se = (1 - a_1^2, -a_1a_2, \ldots, -a_1a_n)$ and $TSe = (0, -a_1a_2, \ldots, -a_1a_n)$, we obtain

$$|H \cap C_n| = \frac{\left((1-a_1^2)^2 + a_1^2\Sigma_2^n a_i^2\right)^{1/2}}{\left(a_1^2\Sigma_2^n a_i^2\right)^{1/2}} = \frac{1}{a_1} < \sqrt{2}.$$

Hence $|H \cap Q_n| \leqslant |H \cap C_n| < \sqrt{2}$.

Assume otherwise that $a_i \leqslant 1/\sqrt{2}$ for all $i$. Since, if some $a_i$ is zero, the problem reduces to that in $\mathbf{R}^{n-1}$, we may assume inductively that $a_i > 0$ for all $i$, and with this assumption the case of equality will follow if we show that $|H \cap Q_n| < \sqrt{2}$ unless $n = 2$ and $a_1 = a_2 = 1/\sqrt{2}$. With $(X_i)_1^n$ independent random variables uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$ and $f$ the continuous p.d.f. of $X = \sum a_i X_i$ as

above, the random variable $a_i X_i$ has characteristic function

$$\phi_i(t) = \left(2\sin \tfrac{1}{2}a_i t\right)/a_i t,$$

and so the characteristic function of $X$ is

$$\phi(t) = \prod_1^n \phi_i(t) = \prod_1^n \frac{2\sin \frac{1}{2}a_i t}{a_i t} = \int_{-\infty}^{\infty} \cos xt \cdot f(x)\, dx.$$

By the standard Fourier inversion formula,

$$\begin{aligned} |H \cap Q_n| = f(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod_1^n \frac{2\sin \frac{1}{2}a_i t}{a_i t}\, dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \prod_1^n \frac{\sin a_i t}{a_i t}\, dt \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \prod_1^n \left|\frac{\sin a_i t}{a_i t}\right| dt. \end{aligned}$$

Set $p_i = a_i^{-2}$ so that $p_i \geqslant 2$ for all $i$ and $\sum p_i^{-1} = 1$. Then by Hölder's inequality with $n$ exponents,

$$\begin{aligned} |H \cap Q_n| &\leqslant \prod_1^n \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \left|\frac{\sin a_i t}{a_i t}\right|^{p_i} dt\right)^{1/p_i} \\ &= \prod_1^n \left(\frac{1}{a_i \pi}\int_{-\infty}^{\infty} \left|\frac{\sin t}{t}\right|^{p_i} dt\right)^{1/p_i}. \end{aligned}$$

By Lemma 3 this gives

$$|H \cap Q_n| \leqslant \prod_1^n \left(\frac{\sqrt{2}}{a_i\sqrt{p_i}}\right)^{1/p_i} = \prod_1^n (\sqrt{2})^{1/p_i} = \sqrt{2},$$

and there is equality only if $p_i = 2$ for all $i$; i.e., $n = 2$ and $a_1 = a_2 = 1/\sqrt{2}$.

PROOF OF LEMMA 3. We divide the proof into two cases, $p \geqslant 4$ and $2 \leqslant p < 4$.

(a) $p \geqslant 4$. Observe that

$$\begin{aligned} \frac{\sin t}{t} &= 1 - \frac{t^2}{6} + \frac{t^4}{120} - \frac{t^6}{7!} + \frac{t^8}{9!} - \cdots \\ &< 1 - \frac{t^2}{6} + \frac{t^4}{120} \quad \text{if } t^2 < 72, \\ e^{-t^2/6} &= 1 - \frac{t^2}{6} + \frac{t^4}{72} - \frac{t^6}{1296} + \frac{t^8}{6^4 \cdot 4!} - \frac{t^{10}}{6^5 \cdot 5!} + \cdots \\ &> 1 - \frac{t^2}{6} + \frac{t^4}{72} - \frac{t^6}{1296} \quad \text{if } t^2 < 30. \end{aligned}$$

Since

$$\frac{t^4}{72} - \frac{t^6}{1296} - \frac{t^4}{120} = \frac{t^4}{180} - \frac{t^6}{1296} \geqslant 0 \quad \text{if } t^2 \leqslant \frac{36}{5}$$

and $36/5 \leqslant \pi^2$, we have

$$0 \leqslant (\sin t)/t \leqslant e^{-t^2/6} \quad \text{if } t^2 \leqslant 36/5.$$

Let $m = 6/\sqrt{5}$. Then

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^p dt &< \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-pt^2/6}\,dt + \frac{2}{\pi}\int_m^{\infty} t^{-p}\,dt \\ &= \frac{\sqrt{6}}{\sqrt{p}\cdot\sqrt{\pi}} + \frac{2}{\pi(p-1)m^{p-1}} \\ &\leqslant \frac{1}{\sqrt{p}}\left(\frac{\sqrt{6}}{\sqrt{\pi}} + \frac{4}{3\pi m^3}\right) \quad \text{for } p \geqslant 4 \\ &< \frac{\sqrt{2}}{\sqrt{p}}.\end{aligned}$$

(b) $2 \leqslant p < 4$. Setting $s = p/2 - 1$, we wish to show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^{1+s} dt \leqslant \frac{1}{\sqrt{1+s}} \quad \text{for } 0 \leqslant s < 1, \tag{1}$$

with equality if and only if $s = 0$.

For $n = 0, 1, 2, \ldots$ define

$$\alpha_n = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-t^2/\pi}\left(1 - e^{-t^2/\pi}\right)^n, \quad \beta_n = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}\left(1 - \frac{\sin^2 t}{t^2}\right)^n dt.$$

It is easily checked that

$$\beta_0 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}\,dt = 1.$$

Now

$$\begin{aligned}\left(\frac{\sin^2 t}{t^2}\right)^{1+s} &= \frac{\sin^2 t}{t^2}\left(1 - \left(1 - \frac{\sin^2 t}{t^2}\right)\right)^s \\ &= \frac{\sin^2 t}{t^2}\sum_0^{\infty}\frac{(-1)^n s(s-1)\cdots(s-n+1)}{n!}\left(1 - \frac{\sin^2 t}{t^2}\right)^n \\ &= \frac{\sin^2 t}{t^2}\left(1 - \sum_1^{\infty}\frac{|s(s-1)\cdots(s-n+1)|}{n!}\left(1 - \frac{\sin^2 t}{t^2}\right)^n\right),\end{aligned}$$

since $(-1)^n s(s-1)\cdots(s-n+1) \leqslant 0$ if $n \geqslant 1$.

Hence by the monotone convergence theorem,

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^{1+s} dt &= \beta_0 - \sum_1^{\infty}\frac{|s(s-1)\cdots(s-n+1)|}{n!}\beta_n \\ &= 1 - \sum_1^{\infty}\frac{|s(s-1)\cdots(s-n+1)|}{n!}\beta_n.\end{aligned}$$

Now we may write the right side of (1) as

$$\frac{1}{\sqrt{1+s}} = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(e^{-t^2/\pi}\right)^{1+s} dt,$$

and analogously to the above argument, we have

$$\frac{1}{\sqrt{1+s}} = 1 - \sum_{1}^{\infty} \frac{|s(s-1)\cdots(s-n+1)|}{n!}\alpha_n.$$

Hence it will suffice to show that $\alpha_n < \beta_n$ for $n \in \mathbf{N}$.

The first six values of $\alpha_n$ and $\beta_n$ (calculated using the residue theorem in the case of $\beta_n$), correct to two decimal places, are shown in Table 1.

TABLE 1

| $n$ | $\alpha_n$ | $\beta_n$ |
|---|---|---|
| 1 | 0.29 | 0.33 |
| 2 | 0.16 | 0.22 |
| 3 | 0.11 | 0.17 |
| 4 | 0.08 | 0.15 |
| 5 | 0.07 | 0.13 |
| 6 | 0.05 | 0.12 |

It is easily checked that for $n \geqslant 7$,

$$\frac{1}{\sqrt{2\pi}\cdot n} < \frac{1}{2\sqrt{\pi}\sqrt{n+1}} - \frac{1}{2en},$$

so to complete the proof it suffices to show that

$$\alpha_n \leqslant \frac{1}{\sqrt{2\pi}\cdot n}, \tag{2}$$

$$\beta_n \geqslant \frac{1}{2\sqrt{\pi}\sqrt{n+1}} - \frac{1}{2en} \quad \text{for } n \geqslant 7. \tag{3}$$

We obtain these estimates as follows.

For $0 < x \leqslant 1$,

$$\begin{aligned}
|\log x| &\geqslant 1 - x + \tfrac{1}{2}(1-x)^2 + \tfrac{1}{3}(1-x)^3 + \tfrac{1}{4}(1-x)^4 \\
&\geqslant 2(1-x)^2 + (1-x) - \tfrac{3}{2}(1-x)^2 + \tfrac{1}{2}(1-x)^4 \\
&= 2(1-x)^2 + \tfrac{1}{2}x^2(1-x)(3-x) \\
&\geqslant 2(1-x)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha_n &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-x^2/\pi}\left(1 - e^{-x^2/\pi}\right)^n dx \\
&= \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}\left(1 - e^{-u^2}\right)^n du \\
&= \frac{1}{\sqrt{\pi}}\int_0^1 \frac{(1-x)^n}{\sqrt{|\log x|}}\,dx \leqslant \frac{1}{\sqrt{2\pi}}\int_0^1 (1-x)^{n-1}\,dx \\
&= \frac{1}{\sqrt{2\pi}\cdot n}, \quad \text{proving (2).}
\end{aligned}$$

Now plainly,

$$\begin{aligned}\beta_n &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\left(1 - \frac{\sin^2 t}{t^2}\right)^n dt \\ &\geqslant \frac{2}{\pi}\int_1^{\infty} \frac{\sin^2 t}{t^2}\left(1 - \frac{\sin^2 t}{t^2}\right)^n dt \\ &\geqslant \frac{2}{\pi}\int_1^{\infty} \frac{\sin^2 t}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt.\end{aligned}$$

Observe that $(1/t^2)(1 - 1/t^2)^n$ is increasing for $1 \leqslant t \leqslant \sqrt{n+1}$ and decreasing for $t \geqslant \sqrt{n+1}$.

Hence, if $n \geqslant 6$ so that $\sqrt{n+1} > 1 + \pi/2$,

$$\int_{1+\pi/2}^{\sqrt{n+1}} \frac{\sin^2 t}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt \geqslant \int_1^{\sqrt{n+1}-\pi/2} \frac{\cos^2 t}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt$$

and

$$\int_{\sqrt{n+1}}^{\infty} \frac{\sin^2 t}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt \geqslant \int_{\sqrt{n+1}+\pi/2}^{\infty} \frac{\cos^2 t}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt.$$

So

$$\begin{aligned}&\int_1^{\infty} \frac{\sin^2 t}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt \\ &\qquad \geqslant \frac{1}{2}\int_1^{\infty} \frac{1}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt - \frac{1}{2}\int_{\sqrt{n+1}-\pi/2}^{\sqrt{n+1}+\pi/2} \frac{\cos^2 t}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt \\ &\qquad \geqslant \frac{1}{2}\int_1^{\infty} \frac{1}{t^2}\left(1 - \frac{1}{t^2}\right)^n dt - \frac{\pi}{4}\max_t\left(\frac{1}{t^2}\left(1 - \frac{1}{t^2}\right)^n\right) \\ &\qquad = \frac{1}{2}\int_0^1 (1 - u^2)^n du - \frac{\pi}{4(n+1)}\left(1 - \frac{1}{n+1}\right)^n \\ &\qquad = \frac{4^n}{2(2n+1)}\binom{2n}{n}^{-1} - \frac{\pi}{4n}\left(1 - \frac{1}{n+1}\right)^{n+1} \\ &\qquad \geqslant \frac{\sqrt{\pi}}{4\sqrt{n+1}} - \frac{\pi}{4en}.\end{aligned}$$

Hence

$$\beta_n \geqslant \frac{1}{2\sqrt{\pi}\cdot\sqrt{n+1}} - \frac{1}{2en},$$

which is (3). □

The series comparison technique, used in part (b), above, was employed by Haagerup in his determination of the best constants in Khintchine's inequality [**2**]. We are indebted to J. Lindenstrauss for pointing out the similarity between the $L_p$-integral to be evaluated here and those of Haagerup.

Given the identity

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^{1+s}dt = 1 + \sum_1^{\infty}(-1)^n\frac{s(s-1)\cdots(s-n+1)}{n!}\beta_n, \qquad 0 \leqslant s < 1,$$

one could proceed with the proof of Lemma 3, part (b), in a number of ways. One possibility would be to consider the derivative with respect to $s$, as follows.

Let

$$I(s) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^{1+s}dt, \quad J(s) = \frac{1}{\sqrt{1+s}}.$$

Then

$$\begin{aligned} I'(s) &= \frac{d}{ds}\left(1 + \sum_1^{\infty}(-1)^n\frac{s(s-1)\cdots(s-n+1)}{n!}\beta_n\right) \\ &= -\sum_1^{\infty}\frac{\beta_n}{n} + \sum_2^{\infty}s\cdot\frac{(1-s)\cdots(n-1-s)}{n!}\beta_n\cdot\sum_{k=1}^{n-1}\frac{1}{k-s} \\ &= -\sum_1^{\infty}\frac{\beta_n}{n} + \sum_2^{\infty}\frac{s}{n}\cdot\prod_{j=1}^{n-1}\left(1-\frac{s}{j}\right)\cdot\beta_n\sum_{k=1}^{n-1}\frac{1}{k-s} \\ &\leqslant -\sum_1^{\infty}\frac{\beta_n}{n} + \sum_2^{\infty}\frac{s}{n}\beta_n\cdot(3+\log(n-1)) \quad \text{for } 0 \leqslant s \leqslant \frac{1}{2}. \end{aligned}$$

A simple computation shows that $\beta_n \leqslant 1/\sqrt{n}$ for all $n$, and that the above then gives

$$I'(s) \leqslant -\sum_0^{\infty}\frac{\beta_n}{n} + 10s \quad \text{for } 0 \leqslant s \leqslant \frac{1}{2}.$$

Substituting the values of $\beta_n$ for $1 \leqslant n \leqslant 6$ gives

$$-\sum_0^{\infty}\frac{\beta_n}{n} < -0.58,$$

so

$$I'(s) < -1/2 \leqslant J'(s) \quad \text{for } 0 \leqslant s \leqslant 0.008.$$

This gives $I(s) < J(s)$ for $0 < s \leqslant 0.008$. The fact that $I$ and $J$ are decreasing functions enables us to prove that $I(s) < J(s)$ for $0.008 < s \leqslant 1$, using finitely many iterations of the following scheme. Let $s_0 = 0.008$. Using sufficiently accurate values for $\beta_n$ for sufficiently many $n$, find $t_0$ so that $I(s_0) \leqslant t_0 < J(s_0)$. Choose $s_1$ so that $J(s_1) \geqslant t_0$ and repeat to obtain $t_1, s_2, t_2, s_3, \ldots$. For some $k$ we obtain $s_k > 1$, and the result is proven.

**4. Additional remarks.** Despite the similarity of Theorem 4 to the classical Khintchine inequality, the two results do not seem to be readily deducible from one another. However, Theorem 4 and Lemma 1 together provide a simple analogue of Khintchine's inequality for $0 < p < 2$ for independent, uuniformly distributed random variables.

PROPOSITION 5. *Let $X_1, \ldots, X_n$ be independent random variables, uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$ and $p > 0$. Then*

$$\left\|\sum a_i X_i\right\|_p \geqslant \frac{\sqrt{3}}{\sqrt{2}(p+1)^{1/p}}\left\|\sum a_i X_i\right\|_2 = \frac{1}{2\sqrt{2}(p+1)^{1/p}}\left(\sum a_i^2\right)^{1/2}$$

*for any sequence, $(a_i)_1^n$, of scalars.*

It is natural to ask what is the maximum volume (in terms of $k$ and $n$) of the intersection of $Q_n$ with a $k$-dimensional subspace of $\mathbf{R}^n$. While for $k = 1$ the answer is trivially $\sqrt{n}$ and for $k = n - 1$ the answer is provided by Theorem 3, it is not clear what should be the result for general $k$.

This work will form part of a doctoral thesis being written under the supervision of Dr. B. Bollobás, whose constant advice and encouragement have been invaluable.

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