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TOTALLY REAL SUBMANIFOLDS WITH NONNEGATIVE SECTIONAL CURVATURE

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ABSTRACT. We prove that an n-dimensional compact totally real submanifold immersed in an n-dimensional complex space form with parallel mean curvature vector and nonnegative sectional curvature has parallel second fundamental form. Combining our result and Naitoh's works we obtain the classification of such submanifolds.

Introduction. Let M be an n-dimensional compact totally real submanifold immersed in an n-dimensional complex space form of constant holomorphic sectional curvature c. We denote by K the sectional curvature of M. By K. Ogiue [**7**], B. Y. Chen and C. S. Houh [**1**] it is known that if $c > 0$, $K \geqslant (n-2)c/4(2n-1)$ and M is minimal, then either M is totally geodesic or $n = 2$ and M is a flat surface with parallel second fundamental form. B. Y. Chen, C. S. Houh and H. S. Lue [**2**] studied totally real submanifolds with nonnegative sectional curvature and parallel mean curvature vector. Recently, using a method different from them, F. Urbano [**8**] proved that if $c > 0$, $K > 0$ and M is minimal, then M is totally geodesic. On the other hand, H. Naitoh [**5**, **6**] classified completely totally real submanifolds immersed in complex space forms with parallel second fundamental form.

In this note, using the method of A. Gray [**3**, **4**], we prove the following

THEOREM. *Let M be an n-dimensional compact totally real submanifold immersed in an n-dimensional complex space form with parallel mean curvature vector. If M has nonnegative sectional curvature, then the second fundamental form of M is parallel.*

1. Preliminaries. First we review the method of A. Gray. We use the same notation as in [**3**, **4**]. Let M be an n-dimensional Riemannian manifold and denote by R, K and ρ the curvature tensor, sectional curvature and Ricci tensor of M, respectively. We define

$$S(M) = \{(m, x) \mid m \in M, x \in M_m, \|x\| = 1\}, \qquad S_m = \{x \in M_m \mid \|x\| = 1\}.$$

Then $S(M)$ is the unit sphere bundle of M. For $x \in S_m$ we take an orthonormal basis $\{e_1, \ldots, e_n\}$ of M_m such that $x = e_1$. Denote by $(y_2, \ldots, y_n)$ the corresponding system of normal coordinates defined on a neighborhood of x in S_m.

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LEMMA 1. *Let F: $S_m \to \mathbf{R}$ be a C^∞ function. Then*

$$(1.1)\qquad \frac{\partial^{\alpha_2+\cdots+\alpha_n}F}{\partial y_2^{\alpha_2}\cdots\partial y_n^{\alpha_n}}(m,x) = \frac{\partial^{\alpha_2+\cdots+\alpha_n}}{\partial u_2^{\alpha_2}\cdots\partial u_n^{\alpha_n}}F\left((\cos r)x + \left(\frac{\sin r}{r}\right)\sum_{\gamma=2}^{n} u_\gamma e_\gamma\right)(0),$$

where $r^2 = \sum_{\gamma=2}^{n} u_\gamma^2$.

We lift the frame $\{e_1,\dots,e_n\}$ to an orthonormal basis $\{f_1,\dots,f_n,g_2,\dots,g_n\}$ of the tangent space $S(M)_{(m,x)}$. Here we require that $f_1,\dots,f_n$ are horizontal and $g_2,\dots,g_n$ are vertical. Denote by $(x_1,\dots,x_n,y_2,\dots,y_n)$ the corresponding normal coordinate system on a neighborhood of (m,x) in $S(M)$. We define a second order linear differential operator $L(\lambda,\mu)$ on $S(M)$ by

$$(1.2)\qquad L(\lambda,\mu) = \left(\sum_{\alpha=1}^{n}\frac{\partial^2}{\partial x_\alpha^2} + \lambda\sum_{\alpha,\beta=2}^{n} p_{\alpha\beta}\frac{\partial^2}{\partial y_\alpha\partial y_\beta} + \mu\sum_{\alpha=2}^{n} q_\alpha\frac{\partial}{\partial y_\alpha}\right)_{(m,x)},$$

where $p_{\alpha\beta}(m,x) = R_{\alpha x\beta x}$ $(= R(e_\alpha,x,e_\beta,x))$, $q_\alpha(m,x) = \rho_{\alpha x}$ $(= \rho(e_\alpha,x))$. This definition is independent of the choice of normal coordinates at (m,x). Hence $L(\lambda,\mu)_{(m,x)}$ is well defined.

For a C^∞ real-valued function f on $S(M)$ we denote by $\operatorname{grad}^v f$ and $\operatorname{grad}^h f$ the vertical and horizontal components of $\operatorname{grad} f$. In the case of $\lambda = -\mu$ we can prove the following integral formula in a way similar to Lemma 11.3 of [**4**].

LEMMA 2. *If M is compact, then, for a C^∞ real-valued function f on $S(M)$, we have*

$$(1.3)\qquad \int_{S(M)}\left\{fL(\lambda,-\lambda)(f)(m,x) + \|\operatorname{grad}^h f\|^2(m,x) + \lambda K_{x(\operatorname{grad}^v f)(x)}\right\}dV = 0.$$

2. Proof of the Theorem. Let M be an n-dimensional totally real submanifold immersed in an n-dimensional complex space form, and denote by B the second fundamental form of M. We define a (0, 3)-tensor field h on M by

$$h_{uvw} = \langle B(u,v), Jw\rangle, \quad \text{for } u,v,w \in M_m.$$

The covariant derivative of h is given by

$$(\nabla_x h)_{uvw} = \langle(\nabla B)(u,v,x), Jw\rangle, \quad \text{for } u,v,w,x \in M_m,$$

and h and ∇h are symmetric tensor fields. Now we define a function f on $S(M)$ by $f(m,x) = h_{xxx}$ for $(m,x) \in S(M)$.

LEMMA 3. *If M has parallel mean curvature vector, then $L(1/3,-1/3)(f) = 0$.*

PROOF. We compute $\sum_{\alpha=1}^{n}(\nabla_{\alpha\alpha}^2 h)(m,x)$. Since M has parallel mean curvature vector, by the Ricci identity we have

$$(2.1)\qquad \begin{aligned}\sum_{\alpha=1}^{n}(\nabla_{\alpha\alpha}^2 h)(m,x) &= \sum_{\alpha=1}^{n}\nabla_{\alpha\alpha}^2 h_{xxx} = \sum_{\alpha=1}^{n}\nabla_{\alpha x}^2 h_{\alpha xx}\\ &= \sum_{\alpha=1}^{n}\left(\nabla_{x\alpha}^2 h_{\alpha xx} + \sum_{\beta=1}^{n} R_{\alpha x\alpha\beta}h_{\beta xx} + 2\sum_{\beta=1}^{n} R_{\alpha xx\beta}h_{\alpha\beta x}\right)\\ &= \sum_{\beta=1}^{n}\rho_{x\beta}h_{\beta xx} - 2\sum_{\alpha,\beta=1}^{n} R_{\alpha x\beta x}h_{\alpha\beta x}.\end{aligned}$$

We must also compute $(\partial^2 f/\partial y_\alpha \partial y_\beta)(m, x)$. By Lemma 1 we have

$$(2.2)\quad \frac{\partial^2 f}{\partial y_\alpha \partial y_\beta}(m, x) = \frac{\partial^2}{\partial u_\alpha \partial u_\beta} f\left((\cos r)x + \left(\frac{\sin r}{r}\right)\sum_{\gamma>1} u_\gamma e_\gamma\right)(0)$$

$$= \frac{\partial^2}{\partial u_\alpha \partial u_\beta}\left\{(\cos r)^3 h_{xxx} + 3(\cos r)^2\left(\frac{\sin r}{r}\right)\sum_{\gamma>1} u_\gamma h_{xx\gamma}\right.$$

$$+ 3(\cos r)\left(\frac{\sin r}{r}\right)^2 \sum_{\gamma,\delta>1} u_\gamma u_\delta h_{x\gamma\delta}$$

$$\left. + \left(\frac{\sin r}{r}\right)^3 \sum_{\gamma,\delta,\nu>1} u_\gamma u_\delta u_\nu h_{\gamma\delta\nu}\right\}(0)$$

$$= -3\delta_{\alpha\beta} h_{xxx} + 6h_{x\alpha\beta}.$$

Similarly,

$$(2.3)\qquad \frac{\partial f}{\partial y_\alpha}(m, x) = 3h_{xx\alpha}.$$

Since $(\partial^2 f/\partial x_\alpha^2)(m, x) = (\nabla^2_{\alpha\alpha} h)(m, x)$, by (2.1)–(2.3) we have

$$\sum_{\alpha=1}^{n} \frac{\partial^2 f}{\partial x_\alpha^2}(m, x)$$

$$= \sum_{\beta=1}^{n} \rho_{x\beta} h_{\beta xx} - 2\sum_{\alpha,\beta=2}^{n} R_{\alpha x\beta x}\left((1/2)\delta_{\alpha\beta} h_{xxx} + (1/6)\frac{\partial^2 f}{\partial y_\alpha \partial y_\beta}(m, x)\right)$$

$$= \sum_{\beta=2}^{n} \rho_{x\beta} h_{\beta xx} - (1/3)\sum_{\alpha,\beta=2}^{n} R_{\alpha x\beta x}\frac{\partial^2 f}{\partial y_\alpha \partial y_\beta}(m, x)$$

$$= -(1/3)\sum_{\alpha,\beta=2}^{n} R_{\alpha x\beta x}\frac{\partial^2 f}{\partial y_\alpha \partial y_\beta}(m, x) + (1/3)\sum_{\beta=2}^{n} \rho_{x\beta}\frac{\partial f}{\partial y_\beta}(m, x).$$

Hence we see that $L(1/3, -1/3)(f) = 0$. Q.E.D.

THEOREM 4 (URBANO). *Let M be an n-dimensional compact totally real minimal submanifold immersed in an n-dimensional complex projective space. If M has positive sectional curvature, then M is totally geodesic.*

PROOF. Since M has positive sectional curvature, $L(1/3, -1/3)$ is elliptic and $L(1/3, -1/3)(f) = 0$. By the maximum principle f is constant on $S(M)$. Since $h_{xxx} = h_{-x-x-x} = -h_{xxx}$, we have $h = 0$. Hence M is totally geodesic. Q.E.D.

PROOF OF THE THEOREM. By (1.3) of Lemma 2 and Lemma 3 we have

$$\int_{S(M)} \left\{ \|\operatorname{grad}^h f\|^2(m,x) + (1/3) K_{x(\operatorname{grad}^r f)(x)} \right\} dV = 0. \tag{2.4}$$

Since all terms on the left-hand side of (2.4) are nonnegative, each must vanish identically. In particular, $\operatorname{grad}^h f$ is identically zero. This is equivalent to saying that h is parallel. Hence the second fundamental form of M is parallel. Q.E.D.

REMARK. Combining our theorem and results of [**5** and **6**], we obtain the classification of n-dimensional compact totally real submanifolds immersed in n-dimensional complex space forms with parallel mean curvature vector and nonnegative sectional curvature.

Let M be an n-dimensional compact Riemannian manifold with nonnegative sectional curvature and denote by $\overline{M}$ an n-dimensional Hermitian space $\mathbf{C}^n$, an n-dimensional complex projective space $P^n(\mathbf{C})$ or an n-dimensional complex hyperbolic space form $H^n(\mathbf{C})$. Let $\psi\colon M \to \overline{M}$ be an isometric immersion with parallel mean curvature vector.

(1) If $\overline{M} = \mathbf{C}^n$, then $\psi(M)$ is congruent to the standard imbedding of one of the following symmetric R-spaces:

(a) $T \cdot S^{n-1}$ $(n \geqslant 1)$,

(b) $U(p)$ $(n = p^2, p \geqslant 3)$,

(c) $U(p)/O(p)$ $(n = p(p+1)/2, p \geqslant 3)$,

(d) $U(2p)/Sp(p)$ $(n = p(2p-1), p \geqslant 3)$,

(e) $T \cdot E_6/F_4$ $(n = 27)$, and

(f) a Riemannian product of manifolds in (a) ~ (e).

(2) If $\overline{M} = P^n(\mathbf{C})$, then M is locally isometric to a symmetric space $M_0 \times M_1 \times \cdots \times M_r$, where M_0 is of Euclidean type, $\dim M_0 \geqslant r-1$ and M_i $(1 \leqslant i \leqslant r)$ is one of the following irreducible symmetric spaces of compact type:

(a) S^n,

(b) $SU(p)$,

(c) $SU(p)/SO(p)$,

(d) $SU(2p)/Sp(p)$, and

(e) E_6/F_4.

Let $\pi\colon S^{2n+1} \to P^n(\mathbf{C})$ be the Hopf fibration. Then $\pi^{-1}(\psi(M))$ is an $(n+1)$-dimensional submanifold in $\mathbf{C}^{n+1}$ of case (1).

(3) If $\overline{M} = H^n(\mathbf{C})$, then M is locally isometric to a symmetric space $M_0 \times M_1 \times \cdots \times M_r$, where M_0 is of Euclidean type, $\dim M_0 \geqslant r$ and M_i $(1 \leqslant i \leqslant r)$ is one of (a) ~ (e) in case (2). Let $\mathbf{C}^{1,n} = \mathbf{C}^1 \times \mathbf{C}^n$ be a complex vector space with a Hermitian form $F(z,w) = -z_0\overline{w}_0 + \sum_{i=1}^n z_i\overline{w}_i$, for $z = (z_0, z_1, \ldots, z_n)$, $w = (w_0, w_1, \ldots, w_n) \in \mathbf{C}^{1,n}$. Put $H^{2n+1}(4/c) = \{z \in \mathbf{C}^{1,n};\ F(z,z) = 4/c\}$ and let $\pi\colon H^{2n+1}(4/c) \to H^n(\mathbf{C})$ be the natural Riemannian submersion. Then we have $\pi^{-1}(\psi(M)) = S^1 \times \tilde{M}$. Here, for some positive numbers r_0, r_1 with $-r_0^2 + r_1^2 = 4/c$, $S^1 = \{z \in \mathbf{C}^1;\ F(z,z) = -r_0^2\}$ and $\tilde{M}$ is an n-dimensional submanifold in $\mathbf{C}^n$ of case (1).

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