KINEMATIC FORMULAS FOR
WEYL'S CURVATURE INVARIANTS
OF SUBMANIFOLDS IN COMPLEX PROJECTIVE SPACE

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ABSTRACT. It is shown in [5] that Weyl's curvature invariants $k_{2p}(M)$ can be expressed by $\gamma_q \wedge F^{m-q}[M]$, where $M$ is a 2m-dimensional Kahler submanifold with compact closure in a space of constant holomorphic curvature, $\gamma_q$ is the qth Chern form of $M$ and $F$ is the Kahler form of $M$. In this paper, we shall show that each $\gamma_q \wedge F^{m-q}[M]$ is expressible in terms of $F$ and $k_{2p}(M)$. Using this result, we get kinematic formulas for $k_{2p}(M)$ from Shifrin's [8] kinematic formulas for Chern classes.

1. Introduction. In this paper we rely on the notation, terminology and results in [3 and 5]. Let $M$ be a Kahler submanifold of complex dimension $m$ with compact closure in a space $N(\lambda)$ of constant holomorphic curvature $4\lambda$ ($\lambda > 0$). Let $F$ be the Kahler form of $M$ and put $K = R^M - R^{N(\lambda)}$ on $M$, where $R^M$ and $R^{N(\lambda)}$ are the curvature tensors of $M$ and $N(\lambda)$, respectively. Then $K$ is a Kahler curvature-like tensor field on $M$. For $p = 1, \ldots, m$, let $k_{2p}(M)$ be the curvature invariant defined for the tensor field $K$ by Weyl in [9] (see also (7.6) in [4]). For $p = 1, \ldots, m$, let $\gamma_p$ be the pth Chern form of $M$. Put $\Gamma_p(M) = (\gamma_p \wedge F^{m-p})[M]$ for $1 \leq p \leq m$ and $\Gamma_0(M) = F^m[M]$. Then we have (see (3.26) and (3.27) in [5] and also Lemma 4.1 in [6]).

Lemmma 1 (A. Gray [5]). For $1 \leq p \leq m$, we have

(1.1) $k_{2p}(M) = \sum_{q=0}^{p} \frac{(2\pi)^p}{(m-p)!} \binom{m-q+1}{p-q} \left( -\frac{\lambda}{\pi} \right)^{p-q} \Gamma_q(M)$.

Conversely we will get

Lemma 2. For $1 \leq p \leq m$, we have

(1.2) $\Gamma_p(M) = \sum_{q=1}^{p} \frac{(m-q)!}{(2\pi)^q} \binom{m-q+1}{p-q} \left( \frac{\lambda}{\pi} \right)^{p-q} k_{2q}(M) + \binom{m+1}{p} \left( \frac{\lambda}{\pi} \right)^{p} F^m[M]$.

Next we state Shifrin's kinematic formulas (see (3.5) and (4.4) in [8]).

Theorem 1 (T. Shifrin [8]). Let $M^m$ and $N^l$ be Kahler submanifolds of complex dimensions $m$ and $l$, respectively, in a Fubini-Study space $N(\lambda)$. Assume

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that $k = m + l - n \geq 1$. Then, for $1 \leq h \leq k$,

\begin{equation}
\int_{U(n+1) \rightarrow g} \Gamma_h(M^m \cap gN^l) \, dg
\end{equation}

\begin{equation}
= \text{Vol}(U(n+1)) \prod_{0 \leq i, 0 \leq j} (-1)^i \left( \frac{n + h - i - j}{n} \right)^{n + h - i - j} \Gamma_i(M^m) \Gamma_j(N^l)
\end{equation}

whenever both sides make sense. Let $G(l, n)$ be the space of projective $l$-planes in $CP^n (= \mathbb{N}(\lambda))$. Then, for $0 \leq h \leq k$,

\begin{equation}
\int_{G(l,n) \ni L} \Gamma_h(M^m \cap L) \, dL
\end{equation}

\begin{equation}
= \text{Vol}(G(l, n)) \prod_{j=0}^h (-1)^{h-j} \left( \frac{n + h - l - j - 1}{h-j} \right)^{n + h - l - j} \Gamma_j(M^m),
\end{equation}

whenever both sides makes sense.

The normalized Kähler form $\omega = \lambda F/\pi$ is used in [8] so that the coefficients in Shifrin's original kinematic formulas are slightly different from those in the above formulas. Using Lemmas 1 and 2, we obtain from Theorem 1 the following kinematic formulas for Weyl's curvature invariants $k_{2p}$.

**Theorem 2.** Let $M^m$ and $N^l$ be Kähler submanifolds with compact closure in a Fubini-Study space $N(\lambda)$ of constant holomorphic curvature $4\lambda$ ($\lambda > 0$). Assume that $k = m + l - n \geq 1$. Then, for $1 \leq p \leq k$,

\begin{equation}
\int_{U(n+1)} k_{2p}(M^m \cap gN^l) \, dg = \text{Vol}(U(n+1)) \prod_{0 \leq a,b} F_{pab} k_a(M^m) k_b(N^l),
\end{equation}

\begin{equation}
\int_{G(l,n)} k_{2p}(M^m \cap L) \, dL = \text{Vol}(G(l, n)) \left( \frac{\lambda}{\pi} \right)^{n-l} \frac{(m-p)!}{(k-p)!} k_{2p}(M^m),
\end{equation}

where

\begin{equation}
f_{pab} = (-1)^p \left( \frac{\lambda}{\pi} \right)^n \frac{(2\sqrt{\lambda})^{2(p-a-b)} (m-a)(l-b)! (n+k-a-b+2)!}{(p-a-b)! (k-p)! (n+k-p+2)!},
\end{equation}

\begin{align*}
F_{pab} &= f_{pab} \text{ for } a \geq 1 \text{ and } b \geq 1, \\
F_{p0b} &= f_{p0b}/m! \text{ for } p \geq b \geq 1, \\
F_{pa0} &= f_{pa0}/l! \text{ for } p \geq a \geq 1, \quad F_{p00} = f_{p00}/m!l!.
\end{align*}

Weyl's curvature invariants have relations with other invariants. In fact, let $\mu_{2p}(K)$ be the Chern's curvature invariants for the curvature-like tensor $K$ (see [2]). Put $(2p-1)!! = (2p-1) \cdot (2p-3) \cdots 3 \cdot 1$. Then it holds that

\begin{equation}
\mu_{2p}(K) = \left( \begin{array}{c} 2m \\ 2p \end{array} \right) (2p-1)!!^{-1} k_{2p}(M).
\end{equation}

Also, $k_{2p}(M)$ are closely related to mean curvatures of $M$ in $N(\lambda)$ (see, for example, [1]).
2. Proof of Lemma 2. We may put \( k_0(M) = (1/m!) \Gamma_0(M) \). Let, for \( 0 \leq p \leq m \) and \( 0 \leq q \leq m \),

\[
a_p = \frac{(m-p)!}{(2\pi)^p} \left(-\frac{\pi}{\lambda}\right)^p k_{2p}(M), \quad b_q = \left(-\frac{\pi}{\lambda}\right)^q \Gamma_q(M).
\]

Then, from (1.1) we have, for \( 0 \leq p \leq m \), \( a_p = \sum_{q=0}^{p} \binom{m+1-q}{p-q} b_q \). Now, we use formula (7a) (on p. 53 in [7]), and get

\[
b_p = \sum_{q=0}^{p} (-1)^{p+q} \binom{m+1-q}{p-q} a_q.
\]

This implies (1.2).

3. Proof of Theorem 2. First we will show (1.5). If we write (1.3) as

\[
\int \Gamma_h(M^m \cap gN^l) \, dg = \sum_{0 \leq i, 0 \leq j} D_{hij} \Gamma_i(M^m) \Gamma_j(N^l),
\]

by means of (2.1) and (2.2), we have, for \( 1 \leq p \leq k \),

\[
\int k_{2p}(M^m \cap gN^l) \, dg = \sum_{h=0}^{p} \sum_{0 \leq j} \sum_{a=0}^{i} \sum_{b=0}^{j} A_{ph} D_{hij} B_{ia} B_{jb} k_{2a}(M^m) k_{2b}(N^l).
\]

Thus we obtain

\[
\text{Vol.}(U(n+1)) F_{pab} = \sum_{h=a+b \leq i, b \leq j} A_{ph} D_{hij} B_{ia} B_{jb}.
\]

Put

\[
f_{pab} = (-1)^p \binom{\lambda}{n} (2\lambda)^{p-a-b} (m-a)! (l-b)! \frac{(k-p)!}{(k-p)} I_p,
\]

\[
I_p = \sum_{h=a+b \leq i, b \leq j} \binom{k-h+1}{p-h} \binom{n+h-i-j}{n} \binom{m-a+1}{i-a} \binom{l-b+1}{j-b}.
\]

Using (1.1)–(1.3), from (3.1) we get, for \( 1 \leq a, 1 \leq b, a+b \leq p \),

\[
F_{pab} = f_{pab}, \quad F_{p0b} = f_{j0b}/m!, \quad F_{pa0} = f_{pa0}/l!, \quad F_{p00} = f_{p00}/m!!.
\]

To calculate \( I_p \), we write them as

\[
I_p = \sum_{h'=0}^{p'} \sum_{t=0}^{h'} \binom{k'-h+1}{p'-h} \binom{n+h-t}{n} \binom{m'+l'+2}{t},
\]

where \( m' = m-a \), \( l' = l-b \), \( k' = k-a-b \) and \( p' = p-a-b \). We use the identity

\[
\sum_{i+j=t} \binom{m'+1}{i} \binom{l'+1}{j} = \binom{m'+l'+2}{t}
\]
to get (3.5). We know the identities (see, for example, [7])

\[
\sum_{h=\ell}^{p'} \binom{k' - h + 1}{p' - h} \binom{n + h - t}{n} = \binom{n + k' - t + 2}{p' - t},
\]

(3.6)

\[
\sum_{t=0}^{p'} \binom{n + k' + 2}{t} \binom{n + k' + 2 - t}{p' - t} = 2^{p'} \binom{n + k' + 2}{p'}.
\]

Hence, from (3.5), it follows that

\[
I_p = 2^{p'} \binom{n + k' + 2}{p'} = 2^{p-a-b} \binom{n + k - a - b + 2}{p - a - b}.
\]

Combining this with (3.2) and (3.4), we get (1.7).

Next, we prove (1.6). We may put, for \(1 \leq p \leq k\),

\[
\int \Gamma_p(M^m \cap L) \, dL = \sum_{q=0}^{p} E_{pq} \Gamma_q(M^m),
\]

\[
\int k_{2p}(M^m \cap L) \, dL = \sum_{q=1}^{p} G_{pq} k_{2q}(M^m).
\]

Using (2.1) and (2.2), we get

(3.8)

\[
G_{pq} = \sum_{h=q}^{p} \sum_{i=q}^{h} A_{ph} E_{hi} B_{iq}.
\]

Assume that \(1 \leq q < p\). Then, using (1.1), (1.2) and (1.4), we get

(3.9)

\[
G_{pq} = \text{Vol}.(G(l, n))(2\lambda)^{p-q} \left(\frac{\lambda}{\pi}\right)^{-l} \frac{(m-q)!}{(k-p)!} H_{pq},
\]

(3.10)

\[
H_{pq} = \sum_{i'=0}^{p'} \sum_{h'=i'}^{p'} (-1)^{p'-i'} \binom{k' - h' + 1}{p' - h'} \binom{n - l + h' - i' - 1}{h' - i'} \binom{m' + 1}{i'},
\]

where \(i' = i - q\), \(p' = p - q\), \(h' = h - q\) and \(m' = m - q\). Similar to (3.6), we have

\[
\sum_{h' = i'}^{p'} \binom{k' - h' + 1}{p' - h'} \binom{n - l + h' - i' - 1}{h' - i'} = \binom{m' - i' + 1}{m' - p' + 1}.
\]

Hence, from (3.10), it follows that

\[
H_{pq} = (-1)^{p'} \sum_{i'=0}^{p'} (-1)^{i'} \binom{m' + 1 - i'}{m' + 1 - p'} \binom{m' + 1}{i'} = 0.
\]

This gives \(G_{pq} = 0\) for \(1 \leq q < p\). Similarly we get \(G_{p0} = 0\). Thus we obtain \(G_{pq} = 0\) for \(0 \leq q < p\). On the other hand, we have

\[
G_{pp} = A_{pp} E_{pp} B_{pp} = \text{Vol}.(G(l, n))(\frac{\lambda}{\pi})^{-l} \frac{(m-p)!}{(k-p)!}.
\]

Thus we proved (1.6).

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