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INVARIANT IDEALS AND BOREL SETS

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ABSTRACT. We investigate the size of the algebra $\mathcal{B}(I)$, where $\mathcal{B}$ is the family of Borel sets and I is a translation invariant ideal of sets of reals. In particular the question whether $\mathcal{B}(I)$ can contain Vitali selectors or even all sets of reals is discussed in connection with the completeness of I and its invariance.

All our considerations refer to the set $\mathbf{R}$ of reals. $\mathbf{Q}$ denotes the set of rationals and $\mathcal{B}$ the algebra of Borel sets. A family I of subsets of a set A is called an ideal on A iff I contains all singletons, does not contain A, and is closed under the operations of finite unions and of taking subsets. An ideal I on A is called uniform iff $X \in I$ whenever $|X| < |A|$ and is called κ-complete iff $\bigcup X \in I$ whenever $X \subset I$ and $|X| < \kappa$; ω_1-complete ideals are called σ-complete. An ideal I on $\mathbf{R}$ is called invariant iff $x + A \in I$ for every $A \in I$ and $x \in \mathbf{R}$. It is called $\mathbf{Q}$-invariant iff $x + A \in I$ for every $A \in I$ and $x \in \mathbf{Q}$. For any ideal I on $\mathbf{R}$, $\mathcal{B}(I)$ denotes the algebra of sets $\{B \triangle N : B \in \mathcal{B}, N \in I\}$. $\mathcal{P}(X)$ denotes the set of all subsets of X.

If I is the ideal of Lebesgue measure zero sets or of meager sets then $\mathcal{B}(I)$ does not contain Vitali selectors (selectors of the family of cosets of $(\mathbf{Q}, +)$ in $(\mathbf{R}, +)$). We investigate the problem if this remains true for more general classes of invariant or $\mathbf{Q}$-invariant ideals. We also discuss the possibility $\mathcal{B}(I) = \mathcal{P}(\mathbf{R})$ in this context.

Our first theorem shows that for some general classes of ideals the algebra $\mathcal{B}(I)$ does not contain Vitali selectors or other specifically constructed sets.

THEOREM 1. (a) *There exists a set X of reals such that $X \notin \mathcal{B}(I)$ for any 2^ω-complete invariant ideal I on $\mathbf{R}$.*

(b) *Assume CH. There exists a set X of reals such that $X \notin \mathcal{B}(I)$ for any σ-complete $\mathbf{Q}$-invariant ideal I on $\mathbf{R}$.*

(c) *Assume CH and Kurepa's hypothesis. For every σ-complete $\mathbf{Q}$-invariant ideal I there exists a Vitali selector $S \notin \mathcal{B}(I)$.*

PROOF. (a) Let $H = \{h_\alpha : \alpha < 2^\omega\}$ be a Hamel basis and S_α the set of those reals in whose H-representation the elements h_β: $\beta < \alpha$ do not appear at all and the element h_α appears with coefficient 1. Hence those sets are pairwise disjoint and S_α is a selector of the family of cosets of the group generated by $\{h_\beta : \beta \leq \alpha\}$. Let $\{B_\alpha : \alpha < 2^\omega\}$ be an enumeration of $\mathcal{B}$. We define $T_\alpha = S_\alpha \setminus B_\alpha$ and $X = \bigcup_{\alpha < 2^\omega} T_\alpha$. For any $\alpha < 2^\omega$ we get

$$X \triangle B = (X \setminus B_\alpha) \cup (B_\alpha \setminus X) \supset T_\alpha \cup (B_\alpha \cap S_\alpha) = S_\alpha.$$

If I is a 2^ω-complete invariant ideal the sets S_α are outside of I, which shows that $X \notin \mathcal{B}(I)$. □

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(b) Take any uncountable family of pairwise almost disjoint functions $f\colon \omega_1 \to \omega$. It gives also an uncountable family $\{S_\alpha\colon \alpha < \omega_1\}$ of Vitali's selectors. Let $\{B_\alpha\colon \alpha < \omega_1\}$ be an enumeration of $\mathcal{B}$. We construct sets $\{T_\alpha\colon \alpha < \omega_1\}$ by putting

$$T_\alpha = (S_\alpha \backslash B_\alpha) \backslash \bigcup_{\beta<\alpha} S_\beta.$$

Let finally $X = \bigcup_{\alpha<\omega_1} T_\alpha$. Hence, for any $\alpha < \omega_1$

$$X \triangle B_\alpha = (X \backslash B_\alpha) \cup (B_\alpha \backslash X) \supset T_\alpha \cup (B_\alpha \cap S_\alpha) \supset S_\alpha \backslash \bigcup_{\beta<\alpha} S_\beta.$$

Take any σ-complete $\mathbf{Q}$-invariant ideal I. Hence $S_\alpha \notin I$ for $\alpha < \omega_1$ and since the set $S_\alpha \cap \bigcup_{\beta<\alpha} S_\beta$ is countable, this shows that $X \triangle B_\alpha \notin I$ for any $\alpha < \omega_1$. Hence $X \notin \mathcal{B}(I)$. □

(c) By Kurepa's hypothesis there exists a family of ω_2 pairwise almost disjoint functions $f\colon \omega_1 \to \omega$. In view of CH this leads to a family $\{S_\alpha\colon \alpha < \omega_2\}$ of pairwise almost disjoint Vitali selectors. Let I be any σ-complete $\mathbf{Q}$-invariant ideal on $\mathbf{R}$. Then $\forall\, \alpha < \omega_2$, $S_\alpha \notin I$ and $\forall\, \alpha, \beta < \omega_2\ \alpha \neq \beta \Rightarrow S_\alpha \cap S_\beta \in I$. Since by CH $\omega_2 > |\mathcal{B}|$, this shows that for some $\alpha < \omega_2$, $S_\alpha \notin \mathcal{B}(I)$. □

The next theorem shows that Vitali selectors may belong to $\mathcal{B}(I)$ or even to I for some translation invariant ideals.

THEOREM 2. (a) *There exists a 2^ω-complete $\mathbf{Q}$-invariant ideal I such that the algebra $\mathcal{B}(I)$ contains a Vitali selector.*

(b) *There exists an invariant ideal I containing all Vitali selectors.*

PROOF. (a) It is enough to construct a nonempty perfect set P such that $q+P$ is disjoint from P for every $q \in \mathbf{Q}$. Then P is a subset of a Vitali selector S and the set $\bigcup_{q\in\mathbf{Q}} q+(S\backslash P)$ generates a 2^ω-complete $\mathbf{Q}$-invariant ideal I such that $S \in \mathcal{B}(I)$.

We construct the set P by induction simultaneously defining a function $f\colon \omega \to \omega$ and a family of closed sets $\{A_n\colon n \in \omega\}$. In the first step we chose the interval $A_1 = [0,1]$ and let $f(1) = 1$. Suppose that 2^{n-1} closed subintervals of $[0,1]$ of length $1/f(n)$ were chosen in the nth step and their union was called A_n. In the $(n+1)$ step we divide each of those intervals into $(n+1)$ equal closed subintervals and take the coinitial one in each case. Then we divide each of thus obtained 2^{n-1} intervals $I_1, \dots, I_{2^{n-1}}$ of length $1/f(n)(n+1)$ into $2^n - 1$ equal closed subintervals $I_i^1, \dots, I_i^{2^n-1}$, $1 \le i \le 2^{n-1}$. Finally we choose 2^n intervals $I_1^1, I_1^3, I_2^5, I_2^7, \dots, I_i^{4i+1}, I_i^{4i+3}, \dots, I_{2^{n-1}}^{2^n-3}, I_{2^{n-1}}^{2^n-1}$ and call their union A_{n+1}.

The descending closed sets $A_n\colon n \in \omega$ are now defined by induction. We put $P = \bigcap_{n\in\omega} A_n$. P is clearly perfect and $(q+P)\cap P = \varnothing$ for any $q \in \mathbf{Q}$ because in the nth step we assured that $(k/nf(n-1)+A_n)\cap A_n = \varnothing$ for any $1 \le k \le nf(n-1)$. □

(b) Let S be a Vitali selector and $x_1, \dots, x_k \notin \mathbf{Q}$, $q_1, \dots, q_k \in \mathbf{Q}$. Consider any coset $x+\mathbf{Q}$, where $x \in S$. We show that the set $A = (\{x_1, \dots, x_k, q_1, \dots, q_k\} + S) \cap (x+\mathbf{Q})$ is finite. Indeed suppose $x_1+s_1 = x+q_1$, $x_1+s_2 = x+q_2$, where $s_1, s_2 \in S$. Hence $s_1 - s_2 = q_1 - q_2 \in \mathbf{Q}$, which implies $s_1 = s_2$. This shows that A has at most $2k$ elements. Hence for any finite family $\{S_1, \dots, S_n\}$ of Vitali selectors and any finite set C of reals the set $\bigcup_{i=1}^n C + S_i$ has a finite intersection with every coset of $(\mathbf{Q}, +)$ in $(\mathbf{R}, +)$. This implies that the family of all Vitali selectors generates an invariant ideal. □

The above results show that the degree of completeness and of invariance enjoyed by an ideal I may have major impact on the size of $\mathcal{B}(I)$. We will now discuss the question whether $\mathcal{B}(I)$ may contain all sets of reals for $\mathbf{Q}$-invariant or invariant ideals I, gradually restricting attention to more and more complete ones.

If no completeness is assumed then I can be a prime ideal and hence $\mathcal{B}(I)$ contains all subsets of $\mathbf{R}$. In this case we can also require that I be uniform. If we restrict attention to $\mathbf{Q}$-invariant ideals, they cannot be prime anymore (see [**3**, Remark 3]) but may have the property $\mathcal{B}(I) = \mathcal{P}(\mathbf{R})$. It suffices to take the ideal generated by singletons and $\mathbf{R}\backslash\mathbf{Q}$. However this ideal is not uniform (cf. Problem 1). The problem if $\mathcal{B}(I)$ may contain all sets of reals for some invariant ideal I remains open (Problem 2).

We next turn attention to σ-complete ideals. In the $\mathbf{Q}$-invariant setting the existence of such ideals I for which $\mathcal{B}(I)$ contains all sets of reals is independent of ZFC.

PROPOSITION 3. *The statement: "There exists a σ-complete $\mathbf{Q}$-invariant ideal I for which $\mathcal{B}(I) = \mathcal{P}(\mathbf{R})$" is independent of ZFC.*

PROOF. Assuming the continuum hypothesis, for every σ-complete ideal I on $\mathbf{R}$ there is a family $\{X_\alpha\colon \alpha < \omega_1\}$ of pairwise disjoint sets outside of I (because I cannot be σ-saturated). This shows that the algebra $\mathcal{P}(\mathbf{R})/I$ has cardinality 2^{ω_1} hence greater than 2^ω, which implies $\mathcal{B}(I) \neq \mathcal{P}(\mathbf{R})$. On the other hand a lemma of Silver (cf. Martin and Solovay [**1**]) says that if Martin's Axiom (MA) is assumed and A is a set of reals of cardinality $< 2^\omega$ then for every $C \subset A$ there is a Borel set C^* s.t. $C = C^* \cap A$. Assume MA+$\rceil$CH and let A be any union of ω_1 cosets of $(\mathbf{Q}, +)$ in $(\mathbf{R}, +)$. Then the σ-complete ideal I generated by $\mathbf{R}\backslash A$ and all singletons is $\mathbf{Q}$-invariant and $\mathcal{B}(I) = \mathcal{P}(\mathbf{R})$. □

REMARK. The ideal I constructed above is not uniform but those ideals are excluded by the following result of Taylor [**4**]: $|\mathcal{P}(\mathbf{R})/I| > 2^\omega$ for all uniform σ-complete ideals I on $\mathbf{R}$. □

Further restriction to σ-complete invariant ideals prohibits $\mathcal{B}(I)$ from containing all sets of reals.

PROPOSITION 4. *For every σ-complete invariant ideal I, $\mathcal{B}(I) \neq \mathcal{P}(\mathbf{R})$.*

PROOF. We follow an argument from Pelc [**2**]. Let $\{h_\alpha\colon \alpha < 2^\omega\}$ be a Hamel basis and for any finite sequence $s = (q_0, \ldots, q_n)$ of rationals let

$$V_s = \{q_0 h_{\alpha_0} + \cdots + q_n h_{\alpha_n}\colon \alpha_0 < \cdots < \alpha_n < 2^\omega\}.$$

By σ-completeness of I, $V_{s_0} \notin I$ for some finite sequence s_0. Let q be any rational which does not appear in s_0. Clearly for any $\alpha < \beta < 2^\omega$

$$(qh_\alpha + V_{s_0}) \cap (qh_\beta + V_{s_0}) = \emptyset.$$

By invariance of I we get a family of 2^ω pairwise disjoint sets outside of I. This shows that the cardinality of the algebra $\mathcal{P}(\mathbf{R})/I$ is 2^{2^ω} and hence $\mathcal{B}(I) \neq \mathcal{P}(\mathbf{R})$. □

Let us finally remark that for 2^ω-complete ideals I, $\mathcal{B}(I)$ cannot contain all sets of reals even if no invariance is assumed. This follows from Taylor's result mentioned above, since 2^ω-complete ideals are uniform.

We close the paper with a list of open problems. Two of them were mentioned before.

Problem 1. Does there exist a uniform $\mathbf{Q}$-invariant ideal I such that $\mathcal{B}(I) = \mathcal{P}(\mathbf{R})$?

Problem 2. Does there exist an invariant ideal I such that $\mathcal{B}(I) = \mathcal{P}(\mathbf{R})$?

The third problem is due to Brzuchowski and Cichoń [**0**] (unpublished). A partial solution to it is given in Theorem 2(a).

Problem 3. Does there exist a σ-complete invariant ideal I such that $\mathcal{B}(I)$ contains a Vitali selector?

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