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IMMERSIONS OF n-MANIFOLDS INTO $(2n-2)$-MANIFOLDS

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ABSTRACT. We study the question of when a map from M^n to N^{2n-2} is homotopic to an immersion.

1. Introduction. In [**5**] we showed that given a map $g\colon M^n \to N^{2n-1}$ there is an immersion f homotopic to g. In this paper we study the same problem when the dimension of N is $2n-2$. In [**5**] we assumed that the manifolds were differentiable but made no assumption about compactness or boundaries. Here, in general there cannot always be an immersion f, and the problem may be much more difficult. Throughout this paper we will assume that the manifolds are differentiable and closed.

From our result above, the results of Corollary 2.6 below, and the solution of the immersion conjecture by R. Cohen [**1**], one might want to conjecture that for some n there always is a solution and for other n there may not be a solution. (It is clear that if $n = 2^i$, there are M, N, and g where g is not homotopic to an immersion.) Our first result is that such a conjecture is not true. We show that, for all n, there are M, N, and g with g not homotopic to an immersion.

E. Thomas [**11** and **12**] has studied this problem and we build on his results. When $n > 4$, we give some sufficient conditions on M, N, and g to deduce that g is homotopic to an immersion. In case M and N have weakly almost complex structures, we can solve the problem completely. In particular, we show that any $g\colon CP^m \to CP^{2m-1}$ is homotopic to an immersion if $m > 2$ and $m \neq 2^r$ for some r. The idea of all the proofs is to look at a "stable" normal bundle for g and ask whether or not it has geometric dimension $n-2$.

When $n = 4$, E. Thomas has already solved the problem completely [**12**]. We study in more detail the types of immersions one can have for CP^2 and $CP^1 \times CP^1$ into CP^3.

The proofs of the theorems are given in §§3–7.

2. Statement of results. We use the following notation. Let ξ be a vector bundle. $W_i(\xi)$ denotes the mod 2 Stiefel-Whitney class and $w_i(\xi)$ denotes the integral Stiefel-Whitney class when it exists. For example, if the dimension of ξ is i, then it is the Euler class.

Let $g\colon M^n \to N^{2n-2}$ be a map. We let $g'\colon M^n \to N^{2n-2} \times R$ be an immersion homotopic to the given map g. Such an immersion exists by the results of [**5**]. If we can prove that $\nu_{g'}$ has geometric dimension $n-2$, then it follows from the results in [**2**] that g is homotopic to an immersion. Thus our technique is to study $\nu_{g'}$ which

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we denote by ν_g. Let $W_i(g) = W_i(\nu_g)$. Let $v(N) = 1 + v_1(N) + \cdots$ be the Wu class for the manifold N.

THEOREM 2.1. *Let $x \in H^1(m)$.*[3] *Then*

$$\langle W_{n-1}(g) \cdot x, [M] \rangle = \left\langle \sum_{i \geq 0} g^*(v_{n-2^i}(N)) \cdot x^2, [M] \right\rangle.$$

We use this technical theorem to show that not all g are homotopic to immersions, in fact for each n there is a g which is not homotopic to an immersion. Thus this question does not seem to depend on $\alpha(n)$ as suggested by the solution of the immersion conjecture [**1**].

COROLLARY 2.2. *Let g be the composite of $\pi\colon RP^{n-1} \times S^1 \to RP^{n-1}$ and $i\colon RP^{n-1} \to RP^{2n-2}$, where π is the projection on the first factor and i is the inclusion. Then g is not homotopic to an immersion.*

E. Thomas [**11**] has studied the problem of when g is homotopic to an immersion. He shows that if $w_{n-1}(g) = 0$ and some further conditions depending on whether $n \equiv 1, 2$, or $3 \bmod 4$, then g is homotopic to an immersion. These further conditions allow one to show that a second obstruction is zero by showing that the indeterminacy is the whole set. In our theorems below, we show that the second obstruction is zero using quite different methods.

THEOREM 2.3. *Assume $n = 6$ or 10 and assume ν_g is a Spin bundle. Then g is homotopic to an immersion.*

THEOREM 2.4. *Assume $n = 14$, 22, or 26 and assume ν_g has a $\mathrm{BO}\langle 8 \rangle$ structure. Then g is homotopic to an immersion.*

We let $n = 2m$ and assume that M and N have weakly almost complex structures.

THEOREM 2.5. *Let $n > 4$. Then g is homotopic to an immersion if and only if*

(a) *m even: $\exists y \in H^{2m-2}(M; Z)$ such that $2 \cdot (\mathrm{Sq}^2 y + c_1(\nu_g) \cdot y) + c_n(\nu_g) = 0 \in H^{2m}(M; Z_4)$.*

(b) *m odd: $c_n(\nu_g) = 0 \in H^{2m}(M)$ $(= H^{2m}(M; Z_2))$.*

The first part of the following corollary was proved by E. Thomas when m is odd [**12**]. (See also Feder [**2**] for a study of a similar problem.)

COROLLARY 2.6. *Let $m > 2$. Then $g\colon CP^m \to CP^{2m-1}$ is homotopic to an immersion if and only if $m \neq 2^r$, or $m = 2^r$ and the degree of g is odd. Furthermore,*[4] *for any holomorphic immersion f, there are projective transformations $p_i\colon CP^i \to CP^i$ for $i = m$ and $2m - 1$ such that $f = p_{2m-1} \cdot c \cdot p_m$, where c is the holomorphic inclusion.*

The above theorems all assume that $n > 4$. We now look at the case $n = 4$. The following theorem is proved by E. Thomas [**12**] in his study of immersions in codimension 2.

[3] Unless stated, all cohomology groups have coefficients in Z_2.

[4] The algebraic geometric part of this result and that in Theorems 2.8 and 2.9 may be known to algebraic geometers.

THEOREM 2.7. *Let $g\colon M^4 \to N^6$ be a map. Suppose that $W_1(M) = 0$ and that $W_1(g) = 0$. Then g is homotopic to an immersion if and only if $w_3(g) = 0$ and there is a $\beta \in H^2(M; Z)$ such that $\beta \equiv W_2(g) \bmod 2$ and $\beta^2 = P_g$. (See also Li* [**4**].)

Let c and $\bar{c}\colon CP^2 \to CP^3$ be the holomorphic and antiholomorphic embeddings given by $c(z_0, z_1, z_2) = (z_0, z_1, z_2, 0)$ and $\bar{c}(z_0, z_1, z_2) = (\bar{z}_0, \bar{z}_1, \bar{z}_2, 0)$.

The first part of the following theorem is in [**11**].

THEOREM 2.8. *$g\colon CP^2 \to CP^3$ is homotopic to an immersion if and only if g is homotopic to c or $\bar{c}$. For any holomorphic immersion f, there are projective transformations $p_2\colon CP^2 \to CP^2$ and $p_3\colon CP^3 \to CP^3$ such that $f = p_3 \cdot c \cdot p_2$.*

We now consider maps $g\colon CP^1 \times CP^1 \to CP^3$. As in Theorem 2.8, we also have some results about holomorphic immersions in this case.

THEOREM 2.9. *Any g is homotopic to an immersion. Furthermore, any holomorphic immersion $f\colon CP^1 \times CP^1 \to CP^3$ is an embedding and there exists holomorphic self-homeomorphisms p of CP^3 and q of $CP^1 \times CP^1$ such that $f = p \cdot \otimes \cdot q$. ($\otimes$ is defined in Lemma* 7.1.)

3. Proofs of 2.1 and 2.2. The proof of Theorem 2.1 follows very closely the proof of Theorem 3.2 in [**5**].

$$\begin{aligned}
\langle W_{n-1}(g) \cdot x, [M]\rangle &= \langle W(g) \cdot x, [M]\rangle = \langle W(M)^{-1} \cdot g^*(W(N)) \cdot x, [M]\rangle \\
&= \langle (1)\chi(\mathrm{Sq}) \cdot g^*(\mathrm{Sq}(v(N))) \cdot x, [M]\rangle \quad \text{because } W(N) = \mathrm{Sq}(v(N)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Wu's formula and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1)\chi(\mathrm{Sq}) = W(\nu(M)) = W(M)^{-1}, \\
&= \langle 1 \cdot \chi(\mathrm{Sq})(\mathrm{Sq}(g^*(v(N))) \cdot x), [M]\rangle \\
&= \langle g^*(v(N)) \cdot \chi(\mathrm{Sq})(x), [M]\rangle, \quad \text{because } \chi(\mathrm{Sq})\mathrm{Sq} = 1 \text{ by the definition of } \chi, \\
&= \left\langle \sum_{i\geq 0} g^*(v_{n-2^i}(N)) \cdot x^{2^i}, [M] \right\rangle, \quad \text{because } \chi(\mathrm{Sq})(x) = \sum_{i\geq 0} x^{2^i} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } x \text{ is a one dimensional class.}
\end{aligned}$$

(See also Proposition 3.1 of [**5**].)

We now prove Corollary 2.2. To show that $W_{n-1}(g) \neq 0$ and hence that v_g does not have geometric dimension $< n-1$, we use Theorem 2.1 with $x \in H^1(S^1)$. Then $\chi(\mathrm{Sq})(x) = x$ and we get $W_{n-1}(g) \cdot x = \pi^* i^*(v_{n-1}(N)) \cdot x = y^{n-1} \otimes x$ because $v_{n-1}(RP^{2n-2}) = y^{n-1}$.

A similar argument shows the following theorem.

THEOREM 3.1. *Let $n = 2^r + 2^s$ with $r > s$. Then the composition of $\pi\colon RP^{2^r} \times RP^{2^s} \to RP^{2^r}$ and $i\colon RP^{2^r} \to RP^{2n-2}$ is not homotopic to an immersion.*

4. Spin and $\mathrm{BO}\langle 8\rangle$**.** We first prove Theorem 2.3. As before we let ν_g be the normal bundle of an immersion M^n into $N^{2n-2} \times R$ and show that ν_g has geometric dimension $n-2$ when ν_g has a Spin structure. (We note that ν_g has a Spin structure

if and only if $g^*(W_1(N)) = W_1(M)$ and $g^*(W_2(N)) + W_1(M)^2 + W_2(M) = 0$.) To prove this we first prove a general theorem about the second obstruction for Spin bundles.

Let ξ be an orientable $(n-1)$-plane bundle with vanishing Euler class. Let $S(\xi)$ be the associated $(n-2)$-sphere bundle. By Massey [**7**] (see also [**8**]), there exists $a \in H^{n-2}(S(\xi); Z)$ such that $H^*(S(\xi)) = H^*(B) \oplus H^*(B) \cdot a$ as a group. To determine the Steenrod algebra structure on $H^*(S(\xi))$, we have the equation $\mathrm{Sq}^i(a) = \phi_i + W_i \cdot a$, where $\phi_i \in H^{n-2+i}(B)/(\mathrm{Sq}^i + W_i \cdot) H^{n-2}(B; Z)$. ϕ_2 is the second obstruction to a cross-section of ξ. Hence if B has dimension at most n and the Euler class and ϕ_2 both vanish, then ξ has geometric dimension $n-2$.

THEOREM 4.1. *If ξ has a Spin structure and $n = 6$ or 10, then the Euler class and ϕ_2 are* 0.

To prove this theorem we need the following lemma.

LEMMA 4.2. $H^*(B\,\mathrm{Spin}(n-1)) = H^*(B\,\mathrm{Spin})/(W_n, W_{n+1}, \ldots,)$ *in dimensions* $< n+1$.

This lemma follows easily from the results of Quillen [**9**].

PROOF OF 4.1. If $n = 6$ or 10, then $H^{n-1}(B\,\mathrm{Spin}(n-1); Z) = 0$ by Lemma 4.2. Hence $\phi_2 \in H^n(B\,\mathrm{Spin}(n-1))$ is defined universally. If $n = 6$, then $H^6(B\,\mathrm{Spin}(5)) = 0$ by 4.2. If $n = 10$, then $H^{10}(B\,\mathrm{Spin}(9))$ has only $W_4 \cdot W_6$. If $\phi_2 = W_4 \cdot W_6$, then $\mathrm{Sq}^2(\phi_2) = \mathrm{Sq}^2\,\mathrm{Sq}^2(a) = \mathrm{Sq}^3\,\mathrm{Sq}^1(a) = 0 = W_6^2 \neq 0$, a contradiction.

In order to prove Theorem 2.4, we now state and prove a similar general theorem about $\mathrm{BO}\langle 8\rangle$-bundles.

THEOREM 4.3. *If ξ has a $\mathrm{BO}\langle 8\rangle$ structure and $n = 14$, 22, or 26, then the Euler class and ϕ_2 are* 0.

The lemma analogous to 4.2 in this case is the following one, and the proof is not difficult. Here $\mathrm{BO}\langle 8\rangle(n-1)$ is the pullback of $\mathrm{BO}(n-1)$ over $\mathrm{BO}\langle 8\rangle$.

LEMMA 4.4. $H^*(\mathrm{BO}\langle 8\rangle) = Z_2[W_i | \alpha(i-1) > 2]$ (*see Stong* [**10**]).

$$H^*(\mathrm{BO}\langle 8\rangle(n-1)) = H^*(\mathrm{BO}\langle 8\rangle)/(W_n, W_{n+1}, \ldots)$$

in dimensions $< n+1$ *if* $\alpha(n-1) > 2$ *and* $\alpha(n) > 2$.

PROOF OF 4.3. If $n = 14$, 22, or 26, then $H^{n-1}(\mathrm{BO}\langle 8\rangle(n-1); Z) = 0$ by Lemma 4.4. Hence $\phi_2 \in H^n(\mathrm{BO}\langle 8\rangle(n-1))$ is defined universally. $H^n(\mathrm{BO}\langle 8\rangle(n-1))$ has generators as follows: none for $n = 14$, $W_8 \cdot W_{14}$ for $n = 22$, and $W_{12} \cdot W_{14}$ for $n = 26$. If $n = 22$, then $W_8 \cdot W_{14} = \mathrm{Sq}^2(W_8 \cdot W_{12})$, and hence $\phi_2 \equiv 0$ mod indeterminancy. If $n = 26$, then $\mathrm{Sq}^2(W_{12} \cdot W_{14}) = W_{14}^2 \neq 0$ and we conclude as in the proof of 4.1.

Theorems 2.3 and 2.4 now follow directly from Theorems 4.1 and 4.3 respectively.

5. Almost complex manifolds. In this section we assume that $g\colon M^n \to N^{2n-2}$, that $n = 2m$, that $n > 4$, and that M and N have weakly almost complex structures (i.e., their stable tangent bundles have a complex structure). In order to prove Theorem 2.5, we will prove the following more general theorem.

THEOREM 5.1. *Let ξ be a complex m-plane bundle over B, where B has dimension $< 2m+1$. Then ξ has real dimension at most $2m-2$ if and only if the Euler class of ξ vanishes and*

(a) *m even: $\exists y \in H^{2m-2}(B;Z)$ such that $2\cdot(\mathrm{Sq}^2(y)+c_1(\xi)\cdot y)+c_m(\xi)=0\in H^{2m}(B;Z_4)$.*

(b) *m odd: $c_m(\xi)=0\in H^{2m}(B)$.*

PROOF. ξ has real dimension at most $2m-2$ if and only if $\xi\colon B\to \mathrm{BSO}$ lifts to C where C is the pullback in the following diagram:

$$\begin{array}{ccc} C & \to & \mathrm{BU} \\ \downarrow & & \downarrow \\ \mathrm{BSO}(2m-2) & \to & \mathrm{BSO}. \end{array}$$

The fibre of $C\to\mathrm{BU}$ is the same as the fibre of $\mathrm{BSO}(2m-2)\to\mathrm{BSO}$ which is P_{2m-2} for a range of dimensions. (Here $P_i = RP^\infty/RP^{i-1}$.) $\pi_{2m-2}(P_{2m-2})=Z$ and $\pi_{2m-1}(P_{2m-2})=Z_4$ if m even and $Z_2\oplus Z_2$ if m odd. We study the Moore-Postnikov system for the map $C\to\mathrm{BU}$.

$$\begin{array}{ccc} E_2 & & \\ \downarrow & & \\ E_1=\mathrm{BU}\times K(Z,2m-2) & \xrightarrow{k^{2m}} & K(\pi_{2m-1}(P_{2m-2}),2m) \\ \downarrow & & \\ \mathrm{BU}\to K(Z,2m-1) & & \end{array}$$

If m even, then $k^{2m}=2\cdot(\mathrm{Sq}^2+c_1\cdot)(\iota_{2m-2})+c_m\in H^{2m}(E_1;Z_4)$. It is easy to see that the term Sq^2 and the term c_m appear in this formula. The fact that $c_1\cdot$ appears follows from the results of [**6**]. If m odd, then $k^{2m}=((c_m)_2,\mathrm{Sq}^2(\iota_{2m-2}))\in H^{2m}(E_1;Z_2\oplus Z_2)$. The theorem now follows.

In order to prove Corollary 2.6, we must calculate $c_m(\nu_g)$, where $g\colon CP^m\to CP^{2m-1}$ has degree k. When m is even, the term $(\mathrm{Sq}^2+c_1(\nu_g)\cdot)y=0$ and so we must show that $c_m(\nu_g)\equiv 0\ (4)$. When m is odd, we must show only that $c_m(\nu_g)\equiv 0\ (2)$. We need the following lemma, the proof of which is due to I. Gessel whom we thank.

LEMMA 5.2. $c_m(\nu_g)=(m,m)(k-1)^m\in Z=H^{2m}(CP^m;Z)$.

PROOF. $c(\nu_g)=g^*((1+x)^{2m})\cdot(1+x)^{-m-1}$. We want the coefficient of x^m in $(1+kx)^{2m}/(1+x)^{m+1}$.

$$\begin{aligned}(1+x+(k-1)x)^{2m}/(1+x)^{m+1} &= \sum(2m-i,i)[(k-1)x]^i(1+x)^{2m-i-m-1}\\ &= \sum(2m-i,i)[(k-1)x]^i(1+x)^{m-i-1}.\end{aligned}$$

Hence the coefficient we wish is the constant term in

$$\begin{aligned}&\sum(2m-i,i)(k-1)^i x^i(1+x)^{m-i-1}/x^m\\ &\quad=\sum(2m-i,i)(k-1)^i(1/(1+x))((1+x)/x)^{m-i}.\end{aligned}$$

If $i>m$, then $x/(1+x)$ has only positive powers of x. If $i<m$, then $(1+x)^{m-i-1}/x^{m-i}$ has only negative powers of x. If $i=m$, then we get the coefficient to be $(m,m)(k-1)^m$.

The following lemma is well known.

LEMMA 5.3. *$2^{\alpha(m)}$ is the highest power of 2 dividing (m, m).*

The first part of Corollary 2.6 now follows from 2.5, 5.2, and 5.3 as $(m, m) \equiv 0$ (4) unless $m = 2^r$.

Finally, we prove the last part of Corollary 2.6. Let $f\colon CP^m \to CP^{2m-1}$ be a holomorphic immersion. Then ν_f is complex and has complex dimension $m-1$. Thus by Lemma 5.2, $k = 1$. We now proceed as in the proof of Theorem 2.8 below.

6. Immersions of CP^2 in CP^3. In this section we prove Theorem 2.8. $W(CP^3) = (1+k)^4$, $W(CP^2) = (1+k')^3$, and $(1+g^*(k))^4 = W(g)\cdot(1+k')^3$. Similarly, the Pontrjagin classes are given by $P(CP^3) = (1+k^2)^4 = 1+4k^2$, $P(CP^2) = (1+k'^2)^3 = 1+3k'^2$, and $g^*(1+4k^2) = (1+P_g)\cdot(1+3k'^2) = 1+P_g+3k'^2$. Hence, $P_g = 4g^*(k^2) - 3k'^2$. Let $g^*(k) = mk'$. Then $P_g = (4m^2-3)k'^2$. It is easy to see that only if $m = \pm 1$ is $4m^2-3$ a square. And if $g^*(k) = \pm k'$, we have $W(g) = 1+k$, i.e., $W_2(g) = k'$. Let $\beta = \pm k'$, then $\beta \equiv k' \bmod 2$, and $\beta^2 = k'^2 = P_g$. By Theorem 2.7, g is homotopic to an immersion if and only if $g \in [c]$ or $[\bar{c}]$.

Now, suppose $f\colon CP^2 \to CP^3$ is a holomorphic immersion. Then by the multiplication formula for Chern classes, we have $(1+c_1(f))\cdot(1+k)^3 = (1+f^*(k))^4$. If $f^*(k) = -k'$, then the above formula gives $c_1(f) = -7k'$. This is impossible, since we already know that $c_1(f) = \pm k'$. Therefore, $f^*(k) = k'$.

By Chow's theorem [**3**], $f(CP^2)$ is an algebraic subvariety of CP^3. $f^*(k^2) = k'^2$ shows that the degree of $f(CP^2)$ is one, hence it is given by a linear equation

$$a_0z_0 + a_1z_1 + a_2z_2 + a_3z_3 = 0.$$

Notice that any holomorphic self-homeomorphism of CP^n is a projective transformation. Therefore, there exist required p_2 and p_3 such that $f = p_3 \cdot c \cdot p_2$ and the proof is complete.

7. Immersions of $CP^1 \times CP^1$ in CP^3. In this section we prove Theorem 2.9. Let $f\colon CP^1 \times CP^1 \to CP^3$ be a holomorphic immersion. Then by the multiplication formula for Chern classes, we have $f^*(1+k)^4 = (1+c_1(f))\cdot(1+2k'+2k''+4k'k'')$, and hence $c_1(f) = (4m-2)k' + (4n-2)k''$ and $3mn - 2m - 2n + 1 = 0$. It is easy to check that this last equation has a unique solution $m = 1$, $n = 1$. Thus $f^*(k) = k' + k''$ and $c_1(f) = 2k' + 2k''$. To conclude the proof of Theorem 2.9, we now need a lemma.

LEMMA 7.1. *For any field K, denote by K^m the m-dimensional vector space over K. Then the tensor product $\bigotimes\colon K^m \times K^n \to K^{m+n}$ gives an embedding $\bigotimes\colon KP^{m-1} \times KP^{n-1} \to KP^{m+n-1}$, where KP^{m-1} denotes the $(m-1)$-dimensional projective space over K.*

PROOF. $(a_1, \ldots, a_m)\otimes(b_1, \ldots, b_n) = (a_1b_1, \ldots, a_1b_n, \ldots, a_mb_1, \ldots, a_mb_n)$ yields a map $\bigotimes\colon KP^{m-1} \times KP^{n-1} \to KP^{m+n-1}$. Suppose $(a_1, \ldots, a_m) \otimes (b_1, \ldots, b_n) = c(a'_1, \ldots, a'_m) \otimes (b'_1, \ldots, b'_n)$, $c \neq 0$, and without loss of generality, we may assume that $a_1 \neq 0$, $b_1 \neq 0$. Then it follows that $a'_1 \neq 0$, and $b'_1 \neq 0$, and

$$(a_1, \ldots, a_m) = cb'_1/b_1(a'_1, \ldots, a'_m)$$

and similarly for the b's. This shows that $\otimes$ is an embedding and the proof is complete.

Now look at $\bigotimes: CP^1 \times CP^1 \to CP^3$. In inhomogeneous coordinates, it takes the form: $(1,z)\otimes(1,w) = (1,w,z,zw)$. Hence the rank of its Jacobian matrix is 2. Therefore, $\otimes$ is a holomorphic embedding.

Let $(u_0,u_1,u_2,u_3) = (z_0w_0, z_1w_1, z_2w_2, z_3w_3)$, and we see that $(u_0,u_1,u_2,u_3) \in \otimes(CP^1 \times CP^1)$ if and only if $u_0u_3 - u_1u_2 = 0$. By choosing suitable homogeneous coordinates, we can write $\otimes(CP^1 \times CP^1)$ in the following form:

$$u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0.$$

If f is a holomorphic immersion $f: CP^1 \times CP^1 \to CP^3$, then by Chow's theorem [**3**] $f(CP^1 \times CP^1)$ is an algebraic subvariety of CP^3. Since $f^*(k) = k' + k''$, we have $f^*(k^2) = (k'+k'')^2 = 2k'k''$. Hence $f(CP^1 \times CP^1)$ is a quadratic hypersurface, and is described by one of the following equations:

(i) $u_0^2 = 0$,
(ii) $u_0^2 + u_1^2 = 0$,
(iii) $u_0^2 + u_1^2 + u_2^2 = 0$,
(iv) $u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0$

for suitable coordinates.

(i) gives the set CP^2. If $f(CP^1 \times CP^1) = CP^2$, then $CP^1 \times CP^1$ is a covering space of CP^2, but this is impossible. (ii) gives the union of two copies of CP^2. Since f is an immersion, f^{-1} (one copy of CP^2) is an open and closed subset of $CP^1 \times CP^1$, hence must be $CP^1 \times CP^1$ itself, but this is also impossible. (iii) gives the set A which is the union of all CP^1 connecting the point $(0,0,0,1)$ and the points in $\{u_0^2 + u_1^2 + u_2^2 = 0\} \subset CP^2$. Looking at a small neighborhood of $(0,0,0,1)$ in A, we see that A cannot be the image of an immersion. Therefore, $f(CP^1 \times CP^1)$ can only be the surface

$$u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0$$

and there exist a projective transformation $p: CP^3 \to CP^3$ and a holomorphic self-homeomorphism q of $CP^1 \times CP^1$ such that $f = p \cdot \otimes \cdot q$. Thus we have proved Theorem 2.9.

We remark that it is known that a holomorphic self-homeomorphism q of $CP^1 \times CP^1$ takes the form $q(x,y) = (f(x), g(y))$ or $(f(y), g(x))$ where f and g are projective transformations of CP^1.

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