A TRANSFER FOR COMPACT LIE GROUP ACTIONS
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ABSTRACT. A short construction of a transfer homomorphism from $H_*(X/G)$ to $H_*(X/H)$, where $H \subseteq G$ are compact Lie groups and $X$ is a $G$-CW-complex, is given.

We give here a quick construction of a transfer homomorphism for compact Lie group actions due to Lewis, May and McClure [2]. More specifically, we sketch a shortened version of their proof of

THEOREM. Let $G$ be a compact Lie group, $H \subseteq G$ a closed subgroup, and $X$ a $G$-CW-complex. Then there exist homomorphisms (for any coefficient group $R$)

$$trf : H_*(X/G; R) \rightarrow H_*(X/H; R) \quad \text{and}$$
$$trf^* : H^*(X/H; R) \rightarrow H^*(X/G; R),$$

natural in $X$, and such that $\pi_* \circ trf$ and $trf^* \circ \pi^*$ are multiplication by $\chi(G/H)$. Here, $\pi : (X/H) \rightarrow X/G$ denotes the projection.

For convenience, we concentrate on the homology transfer—the construction in cohomology is the same. Let $G$ be a fixed compact Lie group. By $G$-representation will always be meant a finite dimensional representation over $\mathbb{R}$. For any $G$-representation $V$, $S^V$ will denote its one-point compactification. If $X$ is a $G$-CW-complex with basepoint $x_0 \in X^G$, then we write $\Sigma^V X = S^V \wedge X$ for its $V$-suspension.

The following “skeletal approximation theorem” for suspensions will be needed.

LEMMA. Let $X, Y$ be pointed $G$-CW-complexes, and let $V$ be a $G$-representation. Then any $G$-map $f : \Sigma^V X \rightarrow \Sigma^V Y$ is $G$-homotopic to a “skeletal” map $f'$; i.e., $f'(\Sigma^V (X^n)) \subseteq \Sigma^V (Y^n)$ for all $n \geq 0$.

PROOF. For each $n$, $\Sigma^V (X^n)$ is constructed by attaching, to $\Sigma^V (X^{n-1})$, pairs of the form

$$(D(V) \times G/H \times D^n, (S(V) \times G/H \times D^n) \cup (D(V) \times G/H \times S^{n-1}))$$

$$\simeq (G/H \times D(V \times \mathbb{R}^n), G/H \times S(V \times \mathbb{R}^n)).$$

Here, $D(-)$ and $S(-)$ denote unit disk and sphere, and “attaching the pair $(X_1, X_2)$” means attaching $X_1$ via a map defined on $X_2$. Furthermore, for any $G$-space $Z$, there is a bijection

$$\text{Map}_G(G/H \times D(V \times \mathbb{R}^n), Z) \simeq \text{Map}_H(D(V \times \mathbb{R}^n), Z)$$

(note that $G/H \times D(V \times \mathbb{R}^n) \simeq G \times_H D(V \times \mathbb{R}^n)$).

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It thus suffices to prove, for any $H \subseteq G$, any $n \geq 0$, any $H$-equivariant map $\varphi: D(V \times \mathbb{R}^n) \to S^V \times Y$, and any $H$-homotopy $\Psi = (\psi_t)$ of $\psi = \varphi|S(V \times \mathbb{R}^n)$ with $\operatorname{Im}(\psi_1) \subseteq S^V \wedge Y^{n-1}$, that $\Psi$ extends to a homotopy $\Phi = (\varphi_t)$ of $\varphi$ such that $\operatorname{Im}(\varphi_1) \subseteq S^V \wedge Y^n$. For each $K \subseteq H$, set

$$d_K = \dim(V \times \mathbb{R}^n)^K = n + \dim(V^K).$$

If the pair $((S^V \wedge Y)^K, (S^V \wedge Y^n)^K)$ is $d_K$-connected for all $K \subseteq H$, then the construction of $\Phi$ is immediate. But $(S^V \wedge Y)^K$ is obtained by attaching, to $(S^V \wedge Y^n)^K$, pairs of the form

$$(D(V)^K \times (G/K_i)^K \times D^{n+j}, (S(V)^K \times (G/K_i)^K \times D^{n+j})$$

$$\cup (D(V)^K \times (G/K_j)^K \times S^{n+j-1}))$$

$$\cong (G/K_i)^K \times (D^{d_K+j}, S^{d_K+j-1})$$

for $K_i \subseteq G$ and $j \geq 1$, so $((S^V \wedge Y)^K, (S^V \wedge Y^n)^K)$ is $d_K$-connected. $\Box$

The main step of the construction is the following “homology desuspension” result.

**Proposition.** Let $V$ be a $G$-representation, $X$ and $Y$ (pointed) $G$-CW-complexes, and $f: \Sigma^V X \to \Sigma^V Y$ a $G$-map. Then there is an induced homomorphism (for any coefficient group $R$) $f^\#: \tilde{H}_*(X/G; R) \to \tilde{H}_*(Y/G; R)$, which is natural in $f$, and has the following properties:

1. $(\Sigma^W f)^\# = f^\#$ for any $G$-representation $W$.
2. $f^\# = (f/G)^*$ if $f: X \to Y$ (i.e., $V = 0$).
3. If $\varphi: S^V \to S^V$ is a $G$-map of (ordinary) degree $k$, then $(\varphi \wedge \operatorname{id}_X)^\#$ is multiplication by $k$.

**Proof.** We may assume (suspending if necessary) that $G$ acts orientably on $V$. Fix a map $f: \Sigma^V X \to \Sigma^V Y$. By the lemma, $f$ is homotopic to a map $f'$ such that $f'(\Sigma^V (X^n)) \subseteq \Sigma^V (Y^n)$ for all $n$.

Let $k = \dim(V)$. If

$$X^n = X^{n-1} \cup_\varphi \left( \coprod_i G/H_i \times D^n \right)$$

and

$$Y^n = Y^{n-1} \cup_\psi \left( \coprod_j G/K_j \times D^n \right)$$

(and $X^{-1}, Y^{-1}$ are the basepoints), then

$$H_{n+k}(\Sigma^V (X^n), \Sigma^V (X^{n-1}); R) \cong H_n(X^n, X^{n-1}; R) \cong \sum_i H_0(G/H_i; R)$$

and

$$H_{n+k}(\Sigma^V (Y^n), \Sigma^V (Y^{n-1}); R) \cong H_n(Y^n, Y^{n-1}; R) \cong \sum_j H_0(G/K_j; R).$$

Hence, dividing out linearly with the action of $\overline{G} = \pi_0(G)$, we can identify

$$R \otimes_{\pi_0(G)} H_{n+k}(\Sigma^V (X^n), \Sigma^V (X^{n-1})) \cong \sum_i R \cong H_n(X^n/G, X^{n-1}/G; R)$$

$$= C_n(X/G; R)$$
and similarly for \( Y \). So \( f' \) induces homomorphisms (for all \( n \geq 0 \))

\[
\tilde{f}_n' = R \otimes_{R[G]} H_{n+k}(f'_n, f'_{n-1}): C_n(X/G; R) \to C_n(Y/G; R).
\]

These clearly commute with the boundary maps and hence induce

\[
f_\# = f'_\#: \tilde{H}_n(X/G; R) \to \tilde{H}_n(Y/G; R).
\]

If \( f'' \) is another skeletal approximation to \( f \), then let \( F \) be a skeleton preserving homotopy from \( f' \) to \( f'' \). Then \( \tilde{F}_n \) (in the above notation) defines a chain homotopy between \( \tilde{f}'_n \) and \( \tilde{f}''_n \). It follows that \( f_\# \) is uniquely defined independently of the choice of \( f' \).

Naturality of \( f_\# \) and conditions (1) and (2) are now clear. If \( \varphi: S^V \to S^V \) is any \( G \)-map, then

\[
(\varphi \wedge \text{id}_X)_* : H_{n+k}(\Sigma^V(X^n), \Sigma^V(X^{n-1})) \to H_{n+k}(\Sigma^V(X^n), \Sigma^V(X^{n-1}))
\]

is multiplication by \( \deg(\varphi) \) for any \( n \geq 0 \); and hence \( (\varphi \wedge \text{id}_X)_\# \) is multiplication by \( \deg(\varphi) \).

The construction of the transfer now follows easily. Fix \( H \subseteq G \), and let \( V \) be any \( G \)-representation having an embedding \( G/H \subseteq V \). Let \( t \) denote the composite

\[
t : S^V \to T(\nu_{G/H}) \subseteq T(\nu_{G/H} \oplus \tau_{G/H}) = G/H^+ \wedge S^V,
\]

where \( T(\nu_{G/H}) \) is the Thom space of the normal bundle, and the first map the Pontrjagin-Thom construction. Note that the composite

\[
S^V \xrightarrow{t} G/H^+ \wedge S^V \xrightarrow{\text{proj}} S^V
\]

has degree \( \chi(G/H) \) (see [1, Theorem 2.4]).

If \( X \) is any (nonbased) \( G \)-CW-complex, set

\[
t_X = t \wedge \text{id}_X : \Sigma^V(X^+) \to G/H^+ \wedge S^V \wedge X^+ \cong \Sigma^V(G/H \times X)^+.
\]

Since \( G/H \times X \) has the \( G \)-homotopy type of a \( G \)-CW-complex (Lemma 1) and \( (G/H \times X)/G \cong X/H \), we can now define the transfer for \( X \) by

\[
\text{trf} = (t_X)_*: H_*(X/G; R) \to H_*(G/H \times X/G; R) = H_*(X/H; R).
\]

Naturality of \( \text{trf} \) follows from the naturality of the homology desuspension. That \( \pi_* \circ \text{trf} \) is multiplication by \( \chi(G/H) \) follows from (3) in the proposition.

**REFERENCES**