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ON STABLY EXTENDED PROJECTIVE MODULES OVER POLYNOMIAL RINGS

MOSHE ROITMAN

ABSTRACT. We prove here that if A is a commutative noetherian ring of Krull dimension d and of finite characteristic prime to $d!$, then stably extended projective $A[X_1, \dots, X_n]$-modules of rank $\geq d/2+1$ are extended from A.

We denote by A a commutative ring with unit. $U_r(A)$ is the set of unimodular rows of length r over A. As in [**3**, §5, p. 34], given u, v in $U_r(A)$ and a subgroup G of $GL_r(A)$, we write $u \sim_G v$ if there exists g in G such that $v = ug$. We abbreviate the notations $u \sim_{GL_r(A)} v$ to $u \sim v$ and $u \sim_{E_r(A)} v$ to $u \sim_E v$. For u, v in $U_r(A)$ we denote by $u \leftrightarrow_{GL_r(A)} v$ or simply $u \leftrightarrow v$ the property: $u \sim (1, 0, \dots, 0)$ if and only if $v \sim (1, 0, \dots, 0)$.

If $\varphi: A \to B$ is a canonical ring homomorphism (such as $A \to A_S$, $A \to A/I$, where S is a multiplicative subset and I is an ideal of A) and $a \in A$, we denote $\varphi(a) = \overline{a}$.

If $f(X)$ is a polynomial in $A[X]$, we denote its leading coefficient by $l(f)$. As usual $A(X)$ denotes the localization of $A[X]$ at the set of monic polynomials. If S is a multiplicative subset of A and $f(X) \in A[X]$, we say that $f(X)$ is unitary in $A_S[X]$ if $\overline{f(X)}$ is unitary in $A_S[X]$, that is, $l(f) = us$ for some $s \in S$ and u invertible in A.

We recall that a finitely generated projective module P over $R = A[X_1, \cdots, X_n]$ is called *stably extended* from A if there exists a finitely generated R-projective module Q extended from A such that $P \oplus Q$ is extended from A or, equivalently, if there exists $m \geq 0$ such that $P \oplus R^m$ is extended from A.

LEMMA 1 (CF. [**12**, COROLLARY 2]). *Let* $(x_0, \dots, x_r) \in U_{r+1}(A)$, $r \geq 2$, *and let t be an element of A which is invertible* mod $(Ax_0 + \cdots + Ax_{r-2})$. *Then* $(x_0, \dots, x_r) \sim_E (x_0, \dots, t^2 x_r)$.

PROOF. Let $\sum_{i=0}^r x_i y_i = 1$. Then by [**8**, Lemma 1] we have

$$(x_0, \dots, x_{r-2}, x_{r-1}, x_r) \sim_E (x_0, \dots, x_{r-2}, y_{r-1}, y_r) \sim_E (x_0, \dots, x_{r-2}, t x_{r-1}, t x_r).$$

Let $tt' \equiv 1 \bmod (Ax_0 + \cdots + Ax_{r-2})$. By Whitehead's lemma we have

$$\begin{aligned}(x_0, \dots, x_{r-2}, t x_{r-1}, t x_r) &\sim_E (x_0, \dots, x_{r-2}, t' t x_{r-1}, t^2 x_r) \\ &\sim_E (x_0, \dots, x_{r-2}, x_{r-1}, t^2 x_r). \quad \square\end{aligned}$$

LEMMA 2 (CF. E.G., [**1**, THÉORÈME 1]). *Let S be a multiplicative subset of A, such that A_S is noetherian of finite Krull dimension d. Let* $(\overline{a}_0, \dots, \overline{a}_r) \in$

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$U_{r+1}(A_S)$, $r > d$. Then there exist $b_i \in A$ $(1 \le i \le r)$ and $s \in S$ such that $s \in A(a_1 + b_1 a_0) + \cdots + A(a_r + b_r a_0)$.

PROOF. Similar to that of [**1**, Théorème 1, §3]. We have to choose elements in A (as a_i'' in [**1**, §3, Lemma 2]) in order to avoid certain prime ideals in A which come from prime ideals in A_S. Finally we obtain $A_S(\overline{a_1 + b_1 a_0}) + \cdots + A_S(\overline{a_r + b_r a_0}) = A_S$. □

LEMMA 3. *Let $f(X)$ be a polynomial in $R = A[X]$ of degree $n > 0$, such that $f(0)$ is invertible in A. Then for any $g(X) \in A[X]$ and natural $k \ge \deg g(X) - \deg f(X) + 1$ there exists $h_k(X) \in A[X]$ of degree $< n$ such that $g(X) = X^k h_k(X) \bmod (Rf(X))$.*

PROOF. Let $f(X) = a_0 + \cdots + a_n X^n$, $g(X) = c_0 + \cdots + c_m X^m$. Let $g(X) - c_0 a_0^{-1} f(X) = X h_1(X)$. Then $g(X) \equiv X h_1(X) \bmod (Rf(X))$ and $\deg h_1(X) < \max(m, n)$. Similarly we obtain $h_2(X)$ such that $h_1(X) \equiv X h_2(X) \bmod (Rf(X))$, $g(X) = X^2 h_2(X) \bmod (Rf(X))$, $\deg h_2(X) < \max(m-1, n)$, etc. In this way the lemma easily follows. □

LEMMA 4. *Let $(x_0, \ldots, x_r) \in U_{r+1}(A)$ and $k \equiv 1 \bmod (r!)$. Then for any $0 \le i \le r$ we have $(x_0, \ldots, x_i, \ldots, x_r) \leftrightarrow (x_0, \ldots, x_i^k, \ldots, x_r)$.*

PROOF. Let $i = 0$. Any unimodular row which contains a $(k-1)$-power of an element in A is completable to an invertible matrix by [**9**, Theorem 2]. It follows by [**9**, Corollary 3.3], that if $(x_0, \ldots, x_r) \sim (1, 0, \ldots, 0)$, then $(x_0^k, \ldots, x_r) \sim (1, 0, \ldots, 0)$. On the other hand if $(x_0^k, \ldots, x_r) \sim (1, 0, \ldots, 0)$, then let $x_0 x_0' \equiv 1 \bmod (Ax_1 + \cdots + Ax_r)$ and so,

$$(x_0, \ldots, x_r) \sim_E (x_0^k x_0'^{k-1}, \ldots, x_r) \sim (1, 0, \ldots, 0)$$

by [**9**, Theorem 2] and [**9**, Corollary 3.3]. □

THEOREM 5 (CF. [**1**, THÉORÈME 3]). *Let A be a commutative noetherian ring of finite Krull dimension d, let $r \ge d/2 + 1$ and assume that A is of finite characteristic prime to $r!$. Let P be a finitely generated projective module of rank r over $R = A[X_1, \ldots, X_n]$ such that $P \oplus R$ is extended from A. Then P is extended from A.*

PROOF. By Quillen's Patching Theorem [**6**, Theorem 1′ or **3**, Chapter 5, §1] we may assume A to be local, so $\operatorname{char} A$ is a power of a prime p. Let P_0 be the A-module $P/(X_1 P + \cdots + X_n P)$. We have to show $P \cong P_0 \otimes_R R$. As p is in the nilradical of R, it is enough to show $P/pP \cong (P_0 \otimes_R R)/p(P_0 \otimes_R R)$ (see e.g. [**3**, Chapter 1, Corollary 1.6]). We have

$$\frac{P_0 \otimes_R R}{p(P_0 \otimes_R R)} \cong \frac{P_0}{pP_0} \otimes_{A/pA} \frac{R}{pR} \quad \text{over } \frac{R}{pR} = \frac{A}{pA}[X_1, \ldots, X_n],$$

which means that the R/pR-module $(P_0 \otimes_R R)/p(P_0 \otimes_R R)$ is extended from the A/pA-module P_0/pP_0. Therefore we have to show that the R/pR-module P/pP is extended from A/pA. Replacing A by A/pA, we assume $\operatorname{char} A = p$. By the Quillen induction (see [**6** or **3**, Chapter 5, §3]) we reduce to the case $n = 1$, $R = A[X]$:

Let $n > 1$ and let S be the set of monic polynomials in $A[X_1]$, $A(X_1) = A[X_1]_s$. Then $\dim A(X_1) = \dim A$ [**3**, Chapter 4, Proposition 1.2] and so by induction the

$A(X_1)[X_2,\dots,X_n]$-module P_s is extended from $A(X_1)$. By Horrocks' theorem (see e.g. [**3**, Chapter 4]), P is extended from A.

We have to prove that $GL_{r+1}(R)$ acts transitively on $U_{r+1}(R)$. Let us call *admissible transformations* of a row $u \in U_{r+1}(R)$ elementary transformations and also transformations of the type

$$(x_0,\dots,x_i,\dots,x_r) \to (x_0,\dots,x_i^k,\dots,x_r), \quad \text{where } k \equiv 1 \bmod (r!).$$

By Lemma 4 it is enough to prove the following

Claim. If $u(X) = (f_0(X),\dots,f_{r+1}(X)) \in U_{r+1}(R)$, $r \geq 2$, then u can be transformed to $(1,0,\dots,0)$ using admissible transformations.

We prove the claim by induction on the number N of nonzero coefficients of the polynomials $f_0(X),\dots,f_r(X)$, starting with $N = 1$. Let $N > 1$. We may assume $\deg f_0 > 0$. Let $l(f_0) = a$. If a is invertible, then $u \sim_E (1,0,\dots,0)$ (see, e.g., [**2**, Chapter III, Corollary 1.4]). In our case the proof is much simpler: As $f_0(X)$ is unitary and A is local, there exist just a finite number of maximal ideals in R which contain f_0, so there exists g in R which does not belong to any such ideal and $f_1 \equiv g \bmod (Rf_2 + \cdots + Rf_r)$. As $Rf_0 + Rg = R$, we conclude $u \sim_E (1,0,\dots,0)$. We assume now that a is not invertible in A. By the inductive assumption with respect to the ring $\overline{A} = A/aA$ and the row $\overline{u}(X)$, we can obtain from $u(X)$ a row $v(X) \equiv (1,0,\dots,0) \bmod (Ra)$ using admissible transformations over R. We can perform such transformations so that at every stage the row contains a polynomial which is unitary in R_a. Indeed, if we have to perform, e.g., the elementary transformation

$$(g_0,\dots,g_r) \xrightarrow{T} (g_0, g_1 + hg_0,\dots,g_r)$$

and g_1 is unitary in R_a, then we replace T by the following two transformations:

$$\begin{aligned}(g_0,g_1,\dots,g_r) &\to (g_0 + aX^m g_1, g_1,\dots,g_r)\\ &\to (g_0 + aX^m g_1, g_1 + h(g_0 + aX^m g_1),\dots,g_r),\end{aligned}$$

where $m > \deg g_0$. We assume now $(f_0,\dots,f_r) \equiv (1,0,\dots,0) \bmod (Ra)$ and f_i is unitary in R_a. If $i > 0$, then replace f_0 by $f_0 + aX^m f_i$, where $m > \deg f_0$; so we assume f_0 is unitary in R_a and also $\deg f_0 > 0$. By Lemma 3 we assume now $f_i = X^{2k} g_i$, where $\deg g_i < \deg f_i$ for $1 \leq i \leq r$. By Lemma 1 we assume $\deg f_i < \deg f_0$ for $1 \leq i \leq r$.

Let $\deg f_0 = m_0$. If $m_0 = 1$, then $f_i \in A$ for $1 \leq i \leq r$. Therefore for sufficiently big q we have $(f_0(X) - f_0(0))^q \in Rf_1 + \cdots + Rf_r$. Choose such q of the form p^n and $q \equiv 1 \bmod (r!)$. Then we perform the admissible transformation $(f_0,\dots,f_r) \to (f_0^q,\dots,f_r)$. As $\operatorname{char} A = p$, we have $f_0^q = f_0(0) + (f_0(X) - f_0(0))^q$, so $(f_0^q,\dots,f_r) \sim_{E_n(R)} (1,0,\dots,0)$.

Assume now $m_0 \geq 2$. We use an argument similar to that in the proof of [**1**, §4, Theorem 3′]. Let $(c_1,\dots,c_{m_0(r-1)})$ be the coefficients of $1, X,\dots,X^{m_0-1}$ in the polynomials $f_2(X),\dots,f_r(X)$. By [**3**, Chapter III, Lemma 1.1], the ideal generated in A_a by $A_a \cap (R_a\overline{f}_0 + R_a\overline{f}_1)$ and the coefficients of $\overline{f}_i$ $(2 \leq i \leq r)$ is A_a. As $m_0(r-1) \geq 2 \cdot d/2 = d > \dim R_a$, by Lemma 2 there exists $(c_1',\dots,c_{m_0(r-1)}) \equiv (c_1,\dots,c_{m_0(r-1)}) \bmod ((Rf_0 + Rf_1) \cap A)$, such that $A_a\overline{c}_1' + \cdots + A_a\overline{c}'_{m_0(r-1)} = A_a$. Assume that we have already $A_a\overline{c}_1 + \cdots + A_a\overline{c}_{m_0(r-1)} = A_a$. By [**1**, §4, Lemma 1(b)], the ideal $Af_0 + Af_2 + \cdots + Af_r$ contains a polynomial $h(X)$ of degree $m_0 - 1$

which is unitary in R_a. Let $l(h) = ua^k$, where u is invertible in A. Using Lemma 1, we achieve by elementary transformations

$$(f_0, f_1, \dots, f_r) \to (f_0, a^{2k} f_1, \dots, f_r)$$
$$\to (f_0, a^{2k} f_1 + (1 - a^k u^{-1} l(f_1)h, \dots, f_r)).$$

Now, $a^{2k} f_1 + (1 - a^k u^{-1} l(f_1))h$ is unitary in R_a, so assume f_1 is unitary in R_a, $\deg f_1 = m_1 < \deg f_0$. By Lemma 1 we assume also $\deg f_i < m_1$ for $2 \le i \le r$. Repeating the argument above we lower the degree of f_1 and obtain finally a row of the form $(f_0, f_1, \dots, f_r) \equiv (1, 0, \dots, 0) \bmod (Ra)$ with f_0 unitary in R_a, $m_0 = \deg f_0 > \deg f_1 = 1$ and $f_i \in A$ for $2 \le i \le r$. Let $l(f_1) = ua^k$, where u is invertible in A, $f_0(X) = 1 - ag(X)$. We have by Lemma 3

$$f_2 \equiv f_2 + 1 - a^{km_0} g^{km_0} \equiv f_2 + 1 - a^{km_0} X^q h(X) \bmod (Rf_0)$$

for some q of the form p^n, $q \equiv 1 \bmod (r!)$ and $h(X)$ of degree $< m_0$. As $\deg h(X) < m_0$, we have $a^{km_0} h(X) \equiv b \bmod (Rf_1)$ for some $b \in A$, so we can obtain $f_2(X)$ of the form $f_2(X) = u + cX^q$ with $c \in A$, u invertible in A. By admissible transformations $(f_0, f_1, \dots, f_r) \to (f_0^q, f_1^q, \dots, f_r)$ we obtain $f_0^q, f_1^q \in A[X^q]$. By Lemma 3 we obtain a row of the type $(c_0, c_1, f_2, c_3, \dots, c_r)$, where $c_i \in A$, so by the argument in the case $m_0 = 1$ we finish the proof. □

Using [**11**, Theorem 1.1] we obtain

COROLLARY 6. *If A is a noetherian ring of dimension d and of characteristic prime to $d!$, then projective stably extended $A[X_1, \dots, X_n]$-modules of rank $\ge d/2+1$ are extended from A.*

COROLLARY 7. *If A is a noetherian ring of dimension 2 and of finite odd characteristic, then projective stably extended $A[X_1, \dots, X_n]$-modules are extended from A (that is, $A[X_1, \dots, X_n]$ is a Hermite ring).*

Finally, we obtain the following particular case of the Bass-Quillen conjecture:

COROLLARY 8. *If A is a noetherian regular ring of dimension d and of characteristic prime to $d!$, then finitely generated projective $A[X_1, \dots, X_n]$-modules of rank $\ge d/2 + 1$ are extended from A.*

Without the assumption on the characteristic, Corollary 6 would strengthen Theorem 1.1 in [**11**] restricted to polynomial rings (see also [**10**, Theorem 7.2]) and Corollaries 7 and 8 would generalize the Murthy-Horrocks Theorem (see, e.g., [**3**, Chapter V, Theorem 3.3]).

It can be shown using [**10**, Theorem 7.2] and the arguments above that if A is noetherian of dimension d and $u \in U_{d+1}(A[X])$, then there exist u_1, u_2 in $U_{d+1}(A[X])$ such that $u \sim_E u_1 \sim_E u_2$, u_1 is of the form $(1 + aX^n, b + aX, c_2, \dots, c_r)$ with $a, b, c_2, \dots, c_r$ in A and u_2 is of the form $(1 + aX^n, 1 + bX^m, c_2, \dots, c_r)$ with $a, b, c_2, \dots, c_r$ in A.

Finally we mention some related results. If we assume A is regular, then projective $A[X_1, \dots, X_n]$-modules are extended under certain additional assumptions not necessarily related to Krull dimension; e.g., by Lindel's theorem [**4**] this holds if A is a regular algebra of essentially finite type over a field (see also [**5**]). For a survey of further results apart from those in [**3**] see [**13**]. For more recent results see Suslin's work in Trudy Mat. Inst. Steklov. **168** (1984) (English translation to appear in Proc. Steklov Inst. Math.). See also [**7**].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, HAIFA 31999, ISRAEL

Current address: Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada K7L 3N6