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ON THE COEFFICIENTS OF p-VALENT FUNCTIONS WHICH ARE POLYNOMIALS OF UNIVALENT FUNCTIONS

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ABSTRACT. We give explicit representations of the coefficients of p-valent functions which are polynomials of univalent functions of the class S. With their help we prove the Goodman conjecture in the special case that $f(z) = [\varphi(z)]^p$, $\varphi(z)$ in S. We also obtain sharp upper bounds for the coefficients of the considered p-valent functions in terms of the coefficients of the two component functions.

The function

$$f(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

is p-valent in $D = \{z \mid |z| < 1\}$ if it assumes no value more than p times for $|z| < 1$. If $p = 1$ let S be the class of all univalent functions (1) with $b_1 = 1$. Let $M(p)$ be the class of all p-valent functions (1) such that $f = P \circ \varphi$, where P is a polynomial of degree q, $1 \leqq q \leqq p$, and $\varphi \in S$, i.e., if

$$P(z) = \sum_{k=1}^{p} a_k z^k \tag{2}$$

and

$$\varphi(z) = \sum_{n=1}^{\infty} c_n z^n \in S \qquad (c_1 = 1), \tag{3}$$

then

$$\sum_{n=1}^{\infty} b_n z^n = \sum_{k=1}^{p} a_k [\varphi(z)]^k. \tag{4}$$

Goodman [1] conjectured that for the coefficients of the p-valent functions (1) the inequalities

$$|b_n| \leqq \sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)!(n^2-k^2)} |b_k| \tag{5}$$

hold for $n > p$. For the class $M(p)$, Goodman [1] and Lyzzaik and Styer [2] have established the following result:

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THEOREM 1 (GOODMAN [1] AND LYZZAIK AND STYER [2]). *If $f \in M(p)$, then for $n > p$ the representations*

$$b_n = \sum_{k=1}^{p} E(\varphi, p, k, n) b_k \tag{6}$$

hold, where

$$E(\varphi, p, k, n) = \begin{vmatrix} A_n^{(k)} & A_n^{(k+1)} & \cdots & A_n^{(p)} \\ A_{k+1}^{(k)} & A_{k+1}^{(k+1)} & \cdots & A_{k+1}^{(p)} \\ \vdots & \vdots & & \vdots \\ A_p^{(k)} & A_p^{(k+1)} & \cdots & A_p^{(p)} \end{vmatrix} \tag{7}$$

where the $A_n^{(m)}$ are determined by the expansion

$$[\varphi(z)]^m = \sum_{n=1}^{\infty} A_n^{(m)} z^n. \tag{8}$$

Now we shall give an explicit representation of $A_n^{(m)}$ and with its help we shall obtain some new results for the class $M(p)$.

THEOREM 2. *The coefficients $A_n^{(m)}$ in (8) are given by*

$$A_n^{(m)} = 0, \qquad 1 \leqq n \leqq m - 1 \ (m \geqq 2), \tag{9}$$

and

$$A_n^{(m)} = m! C_{nm}(c_1, \dots, c_{n-m+1}), \qquad n \geqq m \ (m \geqq 1), \ c_1 = 1, \tag{10}$$

where $C_{nm}(c_1, \dots, c_{n-m+1})$ are homogeneous isobaric polynomials of degree k and weight n, namely

$$C_{nm}(c_1, \dots, c_{n-m+1}) = \sum \frac{(c_1)^{\nu_1} \cdots (c_{n-m+1})^{\nu_{n-m+1}}}{\nu_1! \cdots \nu_{n-m+1}!}, \tag{11}$$

where the sum is taken over all nonnegative integers $\nu_1, \dots, \nu_{n-m+1}$ satisfying

$$\nu_1 + \nu_2 + \cdots + \nu_{n-m+1} = m, \quad \nu_1 + 2\nu_2 + \cdots + (n - m + 1)\nu_{n-m+1} = n. \tag{12}$$

PROOF. According to the results in our papers [3 and 4] we have the equality

$$\left\{ \left(\sum_{\nu=1}^{\infty} c_\nu z^\nu \right)^m \right\}_n = m! C_{nm}(c_1, \dots, c_{n-m+1}) \tag{13}$$

keeping in mind (11)–(12) where $\{\cdots\}_n$ denotes the coefficient of z^n in the mth power of the series (3). Hence the equality (13) expresses the assertions of Theorem 2.

REMARK. According to our papers [3 and 4] the polynomials (11) are easily computed with the help of the recursion relation

$$\begin{aligned} C_{nm} &= \frac{1}{m} \sum_{\mu=1}^{n-m+1} c_\mu C_{n-\mu, m-1}, \qquad 1 \leqq m \leqq n, \ n \geqq 1, \ c_1 = 1, \\ C_{nm} &\equiv C_{nm}(c_1, \dots, c_{n-m+1}), \\ C_{n0} &= 0, \quad C_{00} = 1, \quad C_{n1} = c_n, \quad C_{nn} = c_1^n / n!. \end{aligned} \tag{14}$$

See [3] for a brief table of these polynomials.

COROLLARY 1. *If $f \in M(2)$, then we have*

$$b_n = b_1\left(c_n - c_2\sum_{\mu=1}^{n-1} c_\mu c_{n-\mu}\right) + b_2\sum_{\mu=1}^{n-1} c_\mu c_{n-\mu}, \qquad n = 3, 4, \ldots. \tag{15}$$

PROOF. For $p = 2$ from Theorems 1 and 2 and the Remark we obtain

$$b_n = b_1 E(\varphi, 2, 1, n) + b_2 E(\varphi, 2, 2, n), \tag{16}$$

where

$$E(\varphi, 2, 1, n) = 2\begin{vmatrix} C_{n1} & C_{n2} \\ C_{21} & C_{22} \end{vmatrix} \tag{17}$$

and

$$E(\varphi, 2, 2, n) = 2C_{n2} = \sum_{\mu=1}^{n-1} c_\mu c_{n-\mu}. \tag{18}$$

Thus from (16)–(18), keeping in mind (14), we obtain (15).

COROLLARY 2. *The Goodman conjecture* (5) *is true in the special case that*

$$f(z) = [\varphi(z)]^p = \sum_{n=p}^{\infty} b_n z^n, \qquad \varphi(z) \in S,\ b_p = 1,\ p = 1, 2, \ldots, \tag{19}$$

i.e., the inequalities

$$|b_n| \leqq \binom{n+p-1}{n-p}, \qquad n = p, p+1, \ldots, \tag{20}$$

hold, where for $n > p$ the equality sign holds only for the Koebe function

$$\frac{z}{(1-\varepsilon z)^2} = \sum_{n=1}^{\infty} n\varepsilon^{n-1} z^n \in S \qquad (|\varepsilon| = 1). \tag{21}$$

PROOF. From Theorems 1 and 2 it follows that the coefficient b_n in (19) is

$$b_n = E(\varphi, p, p, n) = p! C_{np}(c_1, \ldots, c_{n-p+1}), \qquad c_1 = 1. \tag{22}$$

Recently Professor Louis de Branges at Purdue University, West Lafayette, Indiana [**5**] proved the Bieberbach conjecture for the class S, i.e.,

$$|c_n| \leqq n, \qquad n = 2, 3, \ldots, \tag{23}$$

with equality only for the Koebe function (21). Now from (22) with the help of (11)–(13), (21) and (23) we obtain successively

$$\begin{aligned} |b_n| &= |E(\varphi, p, p, n)| \leqq p! C_{np}(1, \ldots, n-p+1) \\ &= \left\{\left(\sum_{\nu=1}^{\infty} \nu z^\nu\right)^p\right\}_n = \{z^p(1-z)^{-2p}\}_n \\ &= \binom{n+p-1}{n-p} = E\left(\frac{z}{(1-z)^2}, p, p, n\right). \end{aligned} \tag{24}$$

Hence (24) expresses the assertion (20) with (21).

The following theorem gives sharp upper bounds for the coefficients of the class $M(p)$ in terms of the coefficients of the two component functions (2) and (3).

THEOREM 3. *If $f \in M(p)$, then the inequalities*

$$|b_n| \leqq \sum_{k=1}^{p} \binom{n+k-1}{n-k} |a_k|, \qquad n = 1, 2, \ldots, \tag{25}$$

hold, where the equality sign holds only if the coefficients a_k $(1 \leqq k \leqq n)$ of the polynomial (2) *lie on one and the same ray starting from the origin and the univalent function* (3) *is the Koebe function* (21).

PROOF. If $f \in M(p)$, then we have the formula

$$b_n = \sum_{k=1}^{p} k! C_{nk}(c_1, \ldots, c_{n-k+1}) a_k, \qquad n = 1, 2, \ldots, \; c_1 = 1, \tag{26}$$

which follows from (3)–(4) and (13). Now the inequalities (25) are obtained from the formula (26) with the help of the equality in (24).

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