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ON THE ALGEBRA OF BOUNDED HOLOMORPHIC MARTINGALES

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ABSTRACT. The properties of the algebra H^∞ of bounded holomorphic martingales are further studied following the work of N. Th. Varopoulos [**11**]. The purpose of this paper is to discuss the behavior of weak* closed algebras B with $H^\infty \subsetneqq B \subsetneqq L^\infty$ and the support sets of H^∞-functions. We also give some results on the factorization theorem for H^∞ and extreme points of the unit ball of H^∞.

1. Introduction. Let $(z_1(t): t \geq 0), \ldots, (z_m(t): t \geq 0)$ be m independent complex Brownian motions on a complete probability space (Ω, P) such that

$$P(z_1(0) = \cdots = z_m(0) = 0) = 1.$$

For every $t \geq 0$, $\mathcal{F}(t)$ denotes the σ-field generated by $\{z_j(t): 0 \leq s \leq t;\ j = 1, \ldots, m\}$ and the P-null sets, and $\mathcal{F}$ denotes the σ-field generated by $\bigcup_{t\geq 0} \mathcal{F}(t)$. Let $\mathcal{S}$ be the collection of $(\mathcal{F}(t))$-stopping times. Then for every $T \in \mathcal{S}$, we put $\mathcal{F}(T) = \{A \in \mathcal{F}: A \cap \{T \leq t\} \in \mathcal{F}(t) \text{ for all } t \geq 0\}$.

Let us denote by $H^\infty(\Omega)$ the algebra of bounded $(\mathcal{F}(t))$- martingales $(X_t: t \geq 0)$ which admit an Ito integral representation of the form

$$X_t = X_0 + \sum_{j=1}^{m} \int_0^t \alpha_j(s)\, dz_j(s) \qquad (t \geq 0),$$

where $\alpha_1, \ldots, \alpha_m$ are predictable processes. The algebra $H^\infty(\Omega)$ was introduced by N. Th. Varopoulos [**11**], and it was shown that $H^\infty \equiv \{X_\infty: (X_t: t \geq 0) \in H^\infty(\Omega)\}$ is a weak* Dirichlet algebra on $(\Omega, \mathcal{F}, P)$ (cf. [**11**, Theorem 3.1]). See [**10**] for weak* Dirichlet algebras.

Here we investigate properties of H^∞. The main purpose of this paper is to prove the following theorem which implies some results as corollaries on the factorization theorem for H^∞, extreme points of the unit ball of H^∞ and superalgebras of H^∞.

THEOREM. (i) *Let $T \in \mathcal{S}$ with $0 < T < \infty$ a.s., and let A_T be the weak* closed algebra generated by H^∞ and $L^\infty(\Omega, \mathcal{F}(T), P)$. Then $H^\infty \subsetneqq A_T \subsetneqq L^\infty(\Omega, \mathcal{F}, P)$ and $A_T \cap \overline{A}_T = L^\infty(\Omega, \mathcal{F}(T), P)$ (the bar denotes conjugation, here and always). Obviously H^∞ is not weak* maximal.*

(ii) *For every $A \in \bigcup\{\mathcal{F}(T): T \in \mathcal{S}, T < \infty \text{ a.s.}\}$ and for every two positive numbers α and ε, there exists an $X \in H^\infty$ satisfying the following:*

(a) *$X_R = 0$ a.s. on the complement of A, for an arbitrary $R \in \mathcal{S}$,*

(b) *$0 < |X| \leq \alpha$ a.s. on A,*

(c) *$P(A \setminus \{|X| = \alpha\}) < \varepsilon$.*

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2. Notation. For every subset D of $L^\infty(\Omega, \mathcal{F}, P)$, $[D]$ denotes the weak* closed algebra generated by D in $L^\infty(\Omega, \mathcal{F}, P)$. For $0 < p < \infty$, the closure of H^∞ in $L^p(\Omega, \mathcal{F}, P)$ will be denoted by H^p. Suppose that $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. We put $L^p(\mathcal{G}) = L^p(\Omega, \mathcal{G}, P)$, $H^p(\mathcal{G}) = H^p \cap L^p(\mathcal{G})$ and $L^p = L^p(\mathcal{F})$ $(0 < p \leq \infty)$.

For $X \in L^p$ we write $I(X)$ for the characteristic function of the support set $\operatorname{supp}(X)$ of X.

For every Borel set $G \subset \mathbf{C}$, let us denote by $T(G, j)$ the first time at which the Brownian motion $(z_j(t): t \geq 0)$ escapes from G $(1 \leq j \leq m)$. Then we put $\mathcal{D} = \bigcup\{\mathcal{F}(T(G,j)): G$ is a bounded open subset of $\mathbf{C}$ which contains the origin, and $j = 1, \ldots, m\}$.

3. Proof of the Theorem. (1) Since the conditional expectation $E[\cdot|\mathcal{F}(T)]$ is multiplicative on A_T, the proof of Lemma 3 in [**8**] guarantees that $A_T \cap \overline{A}_T = L^\infty(\mathcal{F}(T))$. Thus (i) is true.

(2) We will construct a bounded holomorphic martingale $X = (X_t)$ with properties (a), (b) and (c) in statement (ii) of the Theorem.

Fix an $(\mathcal{F}_t)$-stopping time T such that $A \in \mathcal{F}(T)$ and $T < \infty$ a.s. Let $T(n) = \inf\{t: |z_1(t)| > n\}$ and $X_t^{(n)} = z_1(T(n) \wedge t)$, $t \geq 0$; $n \in \mathbf{N}$. Define a holomorphic martingale $Y^{(n)} = (Y_t^{(n)})$ by $Y^{(n)} = \int a dX^{(n)}$, where

$$a(s, \omega) = \begin{cases} 0 & (0 \leq s \leq T(\omega)), \\ 1 & (T(\omega) < s) \end{cases} \qquad (\omega \in \Omega).$$

Write $A_1 = A \cap \{T < T(1)\}$ and $A_n = A \cap \{T(n-1) \leq T < T(n)\}$ $(n \geq 2)$, and let

$$R(n) = \begin{cases} T & \text{on } (A_n)^c, \text{ where } (A_n)^c \text{ is the complement of } A_n, \\ \infty & \text{on } A_n \end{cases}$$

$(n \in \mathbf{N})$. Since A_n belongs to $\mathcal{F}(T)$, $R(n)$ is an $(\mathcal{F}_t)$-stopping time. Obviously we have that $\{R(n) > T\} = \{R(n) = \infty\} = A_n \subset \{T(n) > T\}$. This yields that

$$\begin{aligned} P(R(n) > T) &= P(\{z_1(T(n)) \neq z_1(T)\} \cap \{R(n) > T\}) \\ &= P(\{z_1(T(n) \wedge R(n)) \neq z_1(T(n) \wedge T)\} \cap \{R(n) > T\}) \\ &= P(\{X_{R(n)}^{(n)} \neq X_T^{(n)}\} \cap \{R(n) > T\}). \end{aligned}$$

Hence the support set of $Y_{R(n)}^{(n)}$ is A_n $(n \in \mathbf{N})$, that is, $Y_{R(n)}^{(n)} = 0$ a.s. on $(A_n)^c$ and $Y_{R(n)}^{(n)} \neq 0$ a.s. on A_n. By this fact there exists a $\delta_n \in]0, 1[$ with

$$P(A_n \setminus \{|Y_{R(n)}^{(n)}| > \delta_n\}) < (\varepsilon/2^n).$$

Put $S(n) = \inf\{t: |Y_{R(n)\wedge t}^{(n)}| \geq \delta_n\}$. It is easy to check that the support set of $Y_{R(n)\wedge S(n)}^{(n)}$ is A_n $(n \in \mathbf{N})$. Since $A_1, A_2, \ldots$ are mutually disjoint, we get that $\lim_{k\to\infty} \sum_{n=1}^k (\alpha/2^n\delta_n) Y_{R(n)\wedge S(n)}^{(n)}$ exists a.s. and $|\sum_{n=1}^\infty (\alpha/2^n\delta_n) Y_{R(n)\wedge S(n)}^{(n)}| \leq \alpha$ a.s. Hence by the Krein-Smulian consequence (cf. [**2**, Lemma 3.5]), we obtain that

$$X \equiv \sum_{n=1}^\infty (\alpha/2^n\delta_n) Y_{R(n)\wedge S(n)}^{(n)}$$

belongs to H^∞. Further $A \supset \bigcup_{n=1}^\infty A_n$ and $P(A \setminus (\bigcup_{n-1}^\infty A_n)) = 0$, because $\max\{|z_1(t)(\omega)| : 0 \le t \le T(\omega)\} < \infty$ a.s. $\omega \in \Omega$. From this and the definition of X it follows that X satisfies desired properties.

4. Applications of the Theorem. Recently T. Nakazi introduced the following canonical algebras $H^\infty_{\min}$ and $H^\infty_{\max}$ which play an important role in the theory of abstract Hardy algebras [**7**]:

$$H^\infty_{\min} = \bigcap\{B : B \text{ is a weak* closed algebra with } H^\infty \subsetneqq B \subset L^\infty\},$$
$$H^\infty_{\max} = [H^\infty, I(X) : X \in H^\infty].$$

As an application of the Theorem we first describe the canonical algebras in the case of our algebras H^∞.

COROLLARY 1. (1) $H^\infty_{\min} = H^\infty$. (2) $H^\infty_{\max} = L^\infty$.

PROOF. By part (i) of the Theorem and [**4**, p. 79, Lemma 6.1] we have $(\bigcap_{t>0} A_t) \cap (\bigcap_{t>0} A_t)^- = \bigcap_{t>0}(A_t \cap \overline{A}_t) = \bigcap_{t>0} L^\infty(\mathcal{F}(t)) = L^\infty(\mathcal{F}(0)) = \mathbf{C}$. Hence, by [**6**, pp. 516–517] we obtain $\bigcap_{t>0} A_t = H^\infty$, proving Corollary 1(1). Corollary 1(2) is an immediate consequence of part (ii) of the Theorem.

REMARK. Compare the above description of the canonical algebras with the following classical result: Let $H^\infty(\Gamma)$ be the classical Hardy algebra on the unit circle Γ. Then $H^\infty_{\min}(\Gamma) = L^\infty(\Gamma)$ and $H^\infty_{\max}(\Gamma) = H^\infty(\Gamma)$ (see Hoffman [**3**, pp. 194, 52]).

We next consider the factorization theorem of Srinivasan-Wang type for H^∞.

COROLLARY 2. *Suppose* $0 < p < \infty$. *Then for every* $X \in L^p$ *satisfying* $\operatorname{supp}(X) \in \mathcal{D}$ *and* $E[I(X)\log|X|] > -\infty$, *there exist* q, g *and* h *such that* (1) $f = q|g|h$, (2) $|q| = 1$ *a.s. and* $q \in [XH^\infty]_p$, (3) $h \in H^p$ *and is outer, and* (4) $g \in H^\infty$ *and* $|g| = I(X)$ *a.s., where* $[XH^\infty]_p$ *is the* L^p*-closure of* XH^∞.

If $1 \le p < \infty$ and $E[\log|X|] > -\infty$, then Corollary 2 implies the usual factorization of X, because $I(X) = 1$ a.s and $\operatorname{supp}(X) = \Omega \in \mathcal{D}$.

Let $d(\cdot,\cdot)$ be a metric introduced by Gamelin and Lumer, and let H be the universal Hardy class over H^∞ (cf. [**2**, p. 122]). For every $u \in \operatorname{Re}(L^1)$ we denote by ${}^*u$ the conjugate function of u (cf. [**2**, pp. 124, 147]).

PROOF OF COROLLARY 2. Let $f = X + 1 - I(X)$. Define F_n as follows:

$$F_n = \begin{cases} |f| & \text{on } \{(1/n) \le |f| \le n\}, \\ 1/n & \text{on } \{|f| < (1/n)\}, \\ 1 & \text{on } \{n < |f|\}. \end{cases}$$

Then it is easy to check that $\lim_{n\to\infty} \|\log(F_n) - \log|f|\|_1 = 0$. We put $G_n = \exp[\log(F_n) + i^*(\log(F_n))]$ and $K_n = \exp[-\log(F_n) - i^*(\log(F_n))]$. Then $G_n, K_n \in H^\infty$ $(n \in \mathbf{N})$. By Kolmogorov's inequality (cf. [**12**]), there exists a real valued $\mathcal{F}$-measurable function w such that the sequence $\{{}^*(\log(F_n))\}$ converges to w in measure. Since $\lim_{n\to\infty} F_n = f$ a.s., there is a subsequence $\{n(k)\}$ of $\{n\}$ such that $\lim_{k\to\infty}[\log(F_{n(k)}) + i^*(\log(F_{n(k)}))] = \log|f| + iw$ a.s. Let $h = \exp[\log|f| + iw]$ and let $K = \exp[-\log|f| - iw]$. It is not hard to see that $\lim_{k\to\infty} \|G_{n(k)} - h\|_p = 0$ and $\lim_{k\to\infty} d(K_{n(k)}, K) = 0$. Hence $h \in H^p$, $K \in H$ and $hK = 1$. So h is outer. Let $q = Kf$. Since we have easily that $|q| = 1$ a.s. and $\lim_{k\to\infty} \|K_{n(k)}f - q\|_p = 0$,

$q \in [XH^\infty]_p$. By an easy modification of the proof of part (ii) of the Theorem we obtain that there exists $g \in H^\infty$ such that $|g| = I(X)$ a.s. Thus q, h and g are the desired factorization.

From the Theorem we also obtain the following

COROLLARY 3. *Let $T \in \mathcal{S}$ with $T < \infty$ a.s. Let $X \in H^\infty$. Then X_T is an extreme point of the unit ball of H^∞ if and only if $|X_T| = 1$ a.s.*

We omit the proof.

5. Weak* closed algebras which contain H^∞. Let $\tau = \inf\{t: |z_1(t)| \geq 1\}$. Then Corollary 1(1) yields the following

COROLLARY 4. *Let $\mathcal{B}$ be the complete σ-field generated by $z_1(\tau)$. Let B be the weak* closed algebra between H^∞ and L^∞. Then the following are equivalent.*

(1) $B = H^\infty$.

(2) $B \cap \overline{B} \subset L^\infty(\mathcal{B})$.

PROOF. Since the proof of "(1) $\to$ (2)" is obvious, we need prove only "(2) $\to$ (1)". By (2) and [**5**, Proposition 1.4], we have $B \subset [H^\infty, z_1(\tau)^-]$. If $H^\infty \subsetneqq B$, then there exists a weak* closed algebra D such that $H^\infty \subsetneqq D \subsetneqq B \subset [H^\infty, z_1(\tau)^-]$ by Corollary 1(1) [**9**, Corollary 1 and **5**, p. 54]. It is easy to check that $H^\infty(\mathcal{B})$ is a maximal weak* Dirichlet algebra on $(\Omega, \mathcal{B}, P)$. Hence $D \cap L^\infty(\mathcal{B}) = H^\infty(\mathcal{B})$. Since $D \cap \overline{D} \subset B \cap \overline{B} \subset L^\infty(\mathcal{B})$, we have

$$D \cap \overline{D} = (D \cap L^\infty(\mathcal{B})) \cap (D \cap L^\infty(\mathcal{B}))^- = H^\infty(\mathcal{B}) \cap H^\infty(\mathcal{B})^- = \mathbf{C}.$$

By [**6**, pp. 516–517] we obtain $D = H^\infty$. But this contradicts the definition of D. Thus $B = H^\infty$.

COROLLARY 5. *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{B}$. Then the following are equivalent.*

(1) *$E[\cdot|\mathcal{G}]$ is multiplicative on H^∞.*

(2) *$\mathcal{G}$ is a trivial σ-field.*

Hence an operator $N(X)(e^{i\theta}) = E[X|z_1(\tau) = e^{i\theta}]$ is not multiplicative on H^∞.

REMARK. The operator N was introduced by N. Th. Varopoulos (cf. [**11**]). K. Carne [**1**] pointed out that N is not multiplicative on H^∞. He showed more generally that there is no algebra homomorphism from H^∞ to $H^\infty(\Gamma)$. However, Corollary 5 is not a special case of Carne's result, because we do not assume $E[H^\infty|\mathcal{G}] \subset H^\infty$.

PROOF OF COROLLARY 5. Since (2)$\to$(1) is a well-known result (cf. [**11**]), it is sufficient to show the converse (1) $\to$ (2).

Suppose that $E[\cdot|\mathcal{G}]$ is multiplicative on H^∞. Let $B = [H^\infty, L^\infty(\mathcal{G})]$. By the same proof as in Lemma 3 in [**8**] we have $B \cap \overline{B} = L^\infty(\mathcal{G}) \subset L^\infty(\mathcal{B})$. Hence $B = H^\infty$ by Corollary 4. Thus $\mathcal{G}$ is the trivial σ-field.

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