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ON C^m RATIONAL APPROXIMATION[1]

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ABSTRACT. Let $X \subset \mathbf{C}$ be compact and let f be a compactly supported function in $C^m(\mathbf{C})$, $0 < m \in \mathbf{Z}$, such that $\partial f/\partial \bar{z}$ vanishes on X up to order $m-1$. We prove that f can be approximated in $C^m(\mathbf{C})$ by a sequence of functions which are holomorphic in neighborhoods of X.

In this note we prove the following

THEOREM. *Let $X \subset \mathbf{C}$ be compact and let f be a compactly supported continuously differentiable function on $\mathbf{C}$ satisfying*

$$(\partial f/\partial \bar{z})(z) = 0, \qquad z \in X.$$

Then there exists a sequence of continuously differentiable functions h_n, each holomorphic in a neighborhood of X, such that $h_n \to f$ and $\nabla h_n \to \nabla f$ uniformly on $\mathbf{C}$.

This solves a problem raised by O'Farrell in [**7**]. For earlier related work the reader is referred to [**4**, **5**]. See [**1**] and the references given there for analogous problems in several complex variables.

The proof of the theorem is simple. It depends on the Vitushkin constructive scheme for rational approximation and on Nguyen Xuan Uy's theorem [**3**] on the removable sets for holomorphic Lipschitz functions. We have also a higher order version of the theorem whose proof is even more elementary because Nguyen Xuan Uy's theorem is not needed. To state it conveniently we need some notation.

Define the differential operators $\overline{\partial}$ and ∂ by

$$2\overline{\partial} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \quad \text{and} \quad 2\partial = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}.$$

For a positive integer m, $C^m(\mathbf{C})$ is the Banach space consisting of those function on $\mathbf{C}$ with bounded continuous derivatives up to order m. Its norm is

$$\|f\|_m = \sum \|\partial^i \overline{\partial}^j f\|_\infty,$$

the summation being over $0 \le i, j,\ i + j \le m$.

m-THEOREM. *Let $X \subset \mathbf{C}$ be compact and let f be a compactly supported function in $C^m(\mathbf{C})$ satisfying*

$$\partial^i \overline{\partial}^{j+1} f(z) = 0, \qquad z \in X, \quad i + j \le m - 1.$$

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Then there is a sequence of functions h_n in $C^m(\mathbf{C})$, each holomorphic in a neighborhood of X, such that $h_n \to f$ in $C^m(\mathbf{C})$.

PROOF OF THE m-THEOREM, $m \geq 2$. To simplify notation let us consider the case $m = 2$. Fix $\delta > 0$ and consider the δ-Vitushkin scheme for the approximation $(\Delta_j, \varphi_j, f_j)$. In other words, the following holds:

(a) The family (Δ_j) is an almost disjont (each $z \in \mathbf{C}$ belongs to at most 21 Δ_j) countable covering of the plane by open discs Δ_j of radius δ.

(b) $\varphi_j \in C_0^\infty(\Delta_j)$, $0 \leq \varphi_j$, $\sum \varphi_j \equiv 1$, $|\nabla^k \varphi_j| \leq C\delta^{-k}$, $k = 1, 2$. Here and throughout C denotes an absolute constant, unless otherwise specified.

(c)

$$f_j(z) = \frac{1}{\pi} \int \frac{f(\varsigma) - f(z)}{\varsigma - z} \overline{\partial} \varphi_j(\varsigma)\, dm(\varsigma), \qquad z \in \mathbf{C}.$$

It turns out that f_j is holomorphic outside a compact subset of Δ_j, $f_j \equiv 0$ if Δ_j does not intersect spt f and $f = \sum f_j$ (finite sum). See [**2**] for more details.

Let $\sum'$ denote summation over those j such that Δ_j does not intersect X and $\sum''$ summation over the remaining j's. We are going to show that $\sum' f_j$ is a good $C^2(\mathbf{C})$ approximation of f, that is, we will prove that $f - \sum' f_j = \sum'' f_j$ has small $C^2(\mathbf{C})$-norm.

We first estimate $\|\nabla^2 f_j\|_\infty$. We have

$$\overline{\partial} f_j = \varphi_j \overline{\partial} f,$$

$$\partial f_j(z) = \frac{1}{\pi} \int \frac{f(\varsigma) - f(z) - \partial f(z)(\varsigma - z)}{(\varsigma - z)^2} \overline{\partial} \varphi_j(\varsigma)\, dm(\varsigma), \tag{1}$$

$$\overline{\partial}^2 f_j = \overline{\partial} \varphi_j \overline{\partial} f + \varphi_j \overline{\partial}^2 f,$$

$$\partial \overline{\partial} f_j = \partial \varphi_j \overline{\partial} f + \varphi_j \partial \overline{\partial} f,$$

(2)

$$\partial^2 f_j(z) = \text{P.V.} \frac{2}{\pi} \int \frac{f(\varsigma) - f(z) - \partial f(z)(\varsigma - z) - \frac{1}{2}\partial^2 f(z)(\varsigma - z)^2}{(\varsigma - z)^3} \overline{\partial} \varphi_j(\varsigma)\, dm(\varsigma).$$

Assume now that $\Delta_j \cap X \neq \varnothing$. Then

$$\|\overline{\partial}^2 f_j\|_\infty \leq C\omega(\nabla^2 f, \delta), \qquad \|\partial \overline{\partial} f_j\|_\infty \leq C\omega(\nabla^2 f, \delta)$$

because $\overline{\partial} f$, $\overline{\partial}^2 f$ and $\partial \overline{\partial} f$ vanish on X and $|\nabla \varphi_j| \leq C/\delta$.

The analogous estimate for $\|\partial^2 f\|_\infty$ is somewhat subtler. For z in Δ_j Taylor's formula gives

$$|\partial^2 f_j(z)| \leq C|\overline{\partial} f(z)| \left| \text{P.V.} \int \frac{\overline{\varsigma} - \overline{z}}{(\varsigma - z)^3} \overline{\partial} \varphi_j(\varsigma)\, dm(\varsigma) \right| + C\omega(\nabla^2 f, \delta)\delta^{-1} \int_{\Delta_j} |\varsigma - z|^{-1}\, dm(\varsigma).$$

The last term is bounded by a constant times $\omega(\nabla^2 f, \delta)$ because of the well-known inequality $\int_{\Delta_j} |\varsigma - z|^{-1}\, dm(\varsigma) \leq C\delta$. Since $\overline{\partial} f$ vanishes up to order one on X and

Δ_j intersects X, we have $|\bar\partial f(z)| \leq C\omega(\nabla^2 f, \delta)\delta$. On the other hand, if χ denotes the characteristic function of $\Delta(z, 2\delta)$, we get

$$\begin{aligned}\left|\text{P.V.} \int \frac{\bar\varsigma - \bar z}{(\varsigma - z)^3} \bar\partial\varphi_j(\varsigma)\, dm(\varsigma)\right| \\ = \left|\int \frac{\bar\varsigma - \bar z}{(\varsigma - z)^3} (\bar\partial\varphi_j(\varsigma) - \bar\partial\varphi_j(z))\chi(\varsigma)\, dm(\varsigma)\right| \leq C\delta^{-1}.\end{aligned}$$

Thus

$$|\partial^2 f_j(z)| \leq C\omega(\nabla^2 f, \delta), \qquad z \in \Delta_j.$$

It follows from the maximum modulus principle that the last inequality holds for all $z \in \mathbf{C}$. Hence $\|\nabla^2 f_j\|_\infty \leq C\omega(\nabla^2 f, \delta)$. We need now the following lemmas.

LEMMA 1. *Assume h to be holomorphic outside a compact subset of a disc of radius δ, $h \in C^1(\mathbf{C})$ and $h(\infty) = 0$. Then $\|h\|_\infty \leq C\delta\|\nabla h\|_\infty$.*

LEMMA 2. *Let (Δ_j) be an almost disjoint finite sequence of open discs of radius $\delta \leq \frac{1}{2}$, and let $h_j \in C(S^2)$ be holomorphic outside a compact subset of Δ_j. Then*
(i) $\|\sum h_j\|_\infty \leq C\delta^{-1} \max\|h_j\|_\infty$ *if each h_j vanishes at ∞.*
(ii) $\|\sum h_j\|_\infty \leq C \log \delta^{-1} \max\|h_j\|_\infty$ *if each h_j has a double zero at ∞.*
(iii) $\|\sum h_j\|_\infty \leq C \max\|h_j\|_\infty$ *if each h_j has a triple zero at ∞.*
The constants in parts (i) *and* (ii) *depend on the diameter of $\bigcup_j \Delta_j$.*

Lemma 1 is proved in [**6**, p. 193], and Lemma 2(iii) in [**2**, p. 212] (parts (i) and (ii) follow from a similar argument).

We finish the proof of the 2-Theorem as follows:

Since the Δ_j are almost disjoint,

$$\left\|\sum_j{}'' \bar\partial^2 f_j\right\|_\infty \leq C \max\|\bar\partial^2 f_j\|_\infty \leq C\omega(\nabla^2 f, \delta),$$

and similarly $\|\sum'' \partial\bar\partial f_j\|_\infty \leq C\omega(\nabla^2 f, \delta)$.

By Lemma 2(iii),

$$\left\|\sum{}'' \partial^2 f_j\right\|_\infty \leq C \max\|\partial^2 f_j\|_\infty \leq C\omega(\nabla^2 f, \delta).$$

Since the Δ_j are almost disjoint,

$$\left\|\sum{}'' \bar\partial f_j\right\|_\infty \leq C \max\|\bar\partial f_j\|_\infty \leq C\delta\omega(\nabla^2 f, \delta).$$

From Lemma 2(ii) and Lemma 1 we get

$$\left\|\sum{}'' \partial f_j\right\|_\infty \leq C \log \delta^{-1} \max\|\partial f_j\|_\infty \leq C \log \delta^{-1} \cdot \delta\omega(\nabla^2 f, \delta).$$

Finally, from Lemma 2(i) and Lemma 1 we have

$$\left\|\sum{}'' f_j\right\|_\infty \leq C\delta^{-1} \max\|f_j\|_\infty \leq C \max\|\nabla f_j\|_\infty \leq C\delta\omega(\nabla^2 f, \delta).$$

This completes the proof.

If we try the above argument in the case $m = 1$, we would get, after an application of Lemma 2(ii),

$$\left\|\sum{}'' \partial f_j\right\|_\infty \leq C \log \delta^{-1} \cdot \omega(\nabla f, \delta),$$

and the right-hand side tends to zero with δ only if (say) $\omega(\nabla f, \delta)$ satisfies a Dini type condition. To get around this difficulty we use Vitushkin matching coefficient technique and Nguyen Xuan Uy's theorem.

PROOF OF THE 1-THEOREM. Given $\delta > 0$ consider as above the δ-scheme $(\Delta_j, \varphi_j, f_j)$. Fix j with $\Delta_j \cap X \neq \varnothing$ and expand f_j at ∞: $f_j(z) = f_j'(\infty)/z + \cdots$. We have

$$f_j'(\infty) = -\frac{1}{\pi}\int f(\varsigma)\overline{\partial}\varphi_j(\varsigma)\,dm(\varsigma) = \frac{1}{\pi}\int \overline{\partial} f(\varsigma)\varphi_j(\varsigma)\,dm(\varsigma)$$

and so, since $\overline{\partial} f$ vanishes on X,

$$|f_j'(\infty)| \leq C\omega(\nabla f, \delta) m(\Delta_j \backslash X).$$

Nguyen Xuan Uy's theorem asserts that, for all compact $K \subset \mathbf{C}$,

$$m(K) \leq C \sup |h'(\infty)|$$

where the supremum is taken over all functions h in $\operatorname{Lip}(1, \mathbf{C})$, holomorphic outside K and satisfying $\|\nabla h\|_\infty \leq 1$. Then we can find a function $G_j \in \operatorname{Lip}(1, \mathbf{C})$, holomorphic outside a compact subset of $\Delta_j \backslash X$, such that $G_j'(\infty) = 1$ and $\|\nabla G_j\|_\infty \leq C/m(\Delta_j \backslash X)$.

Set $g_j = f_j'(\infty) G_j$, so that $g_j \in \operatorname{Lip}(1, \mathbf{C})$, $g_j'(\infty) = f_j'(\infty)$, $\|\nabla g_j\|_\infty \leq C\omega(\nabla f, \delta)$, and g_j is holomorphic in a neighbourhood of X. regularizing g_j we can assume in addition that $g_j \in C^1(\mathbf{C})$. We claim now that $f - \sum' f_j - \sum'' g_j = \sum'' f_j - g_j$ has small $C^1(\mathbf{C})$ norm. Set $h_j = f_j - g_j$. We have

$$\left\|\overline{\partial}\left(\sum{}'' h_j\right)\right\|_\infty \leq C \max \|\overline{\partial} h_j\|_\infty \leq C\omega(\nabla f, \delta)$$

because the Δ_j are almost disjoint. Estimating the integral in (1) via Taylor's formula, as we did with the integral in (2), we obtain

$$\|\partial h_j\|_\infty \leq \|\partial f_j\|_\infty + \|\partial g_j\|_\infty \leq C\omega(\nabla f, \delta).$$

Notice now that ∂h_j has a triple zero at ∞ because h_j vanishes twice there. Part (iii) of Lemma 2 gives

$$\left\|\partial\left(\sum{}'' h_j\right)\right\|_\infty \leq C \max \|\partial h_j\|_\infty \leq C\omega(\nabla f, \delta).$$

Hence, $\|\nabla(\sum'' h_j)\|_\infty \leq C\omega(\nabla f, \delta)$. From Lemma 2(ii) and Lemma 1 it follows that

$$\left\|\sum{}'' h_j\right\|_\infty \leq C \log \delta^{-1} \max \|h_j\|_\infty \leq C \log \delta^{-1} \cdot \delta\omega(\nabla f, \delta),$$

and this completes the proof.

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