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THE MACKEY CONTINUITY OF THE MONOTONE REARRANGEMENT

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ABSTRACT. Let $(A, \mathcal{A}, \mu)$ be a probability space, and let mes denote the Lebesgue measure on the Borel σ-algebra $\mathcal{B}$ in $[0,1]$. The nondecreasing-rearrangement operator from the space $L^\infty(\mu) = L^\infty(A, \mathcal{A}, \mu)$ of real-valued essentially bounded functions into $L^\infty = L^\infty([0,1], \mathcal{B}, \text{mes})$ is shown to be uniformly continuous in the Mackey topologies $\tau(L^\infty(\mu), L^1(\mu))$ and $\tau(L^\infty, L^1)$ on $L^\infty(\mu)$ and L^∞, respectively.

1. Definitions and notation. Let $(A, \mathcal{A}, \mu)$ be a probability space and let mes denote the Lebesgue measure on the σ-algebra $\mathcal{B}$ of Borel subsets of $[0,1]$. For any real-valued $\mathcal{A}$-measurable function f on A there is a unique, right-continuous, nondecreasing function $f_\uparrow$ on $[0,1]$ such that $\text{mes}(f_\uparrow)^{-1} = \mu f^{-1}$, i.e., $f_\uparrow$ and f are equidistributed. The function $f_\uparrow$ is called the nondecreasing rearrangement of f. The nonincreasing rearrangement $f_\downarrow$ of f is defined as $f_\downarrow = -(-f)_\uparrow$, cf. [**3**, p. 932]. For any $1 \le p \le +\infty$, if $f \in L^p(\mu) = L^p(A, \mathcal{A}, \mu)$, then $f_\uparrow \in L^p = L^p([0,1], \mathcal{B}, \text{mes})$. The term "rearrangement operator" is always used to mean the nondecreasing-rearrangement operator $\uparrow$.

The Hardy-Littlewood-Pólya spectral order $\prec$ is defined as follows: $f \prec g$ if and only if $\int_t^1 f_\uparrow d\,\text{mes} \le \int_t^1 g_\uparrow d\,\text{mes}$ for all $t \in [0,1]$ with equality for $t = 0$.

For a pair of vector spaces E and F in duality, let $\sigma(E,F)$ and $\tau(E,F)$ denote, respectively, the weak and Mackey topologies on E induced by F. The duality between $L^p(\mu)$ and $L^{p'}(\mu)$, where $p^{-1} + (p')^{-1} = 1$, is $\langle f, g\rangle_\mu = \int_A fg\,d\mu$, and the subscript μ is suppressed when $\mu = \text{mes}$.

2. The Mackey continuity of the rearrangement operator. For $1 \le p < +\infty$, the rearrangement operator $\uparrow$ is a nonexpansive mapping of $L^p(\mu)$ into L^p (with their usual norms). This result is contained in [**3**, Corollary (5.2)] with $\phi(x,y) = -|x-y|^p$; note also [**3**, Remark (5.7)]. Alternatively, it can be seen from the relation $f_\uparrow - g_\uparrow \prec f - g$ [**3**, (6.1)(ii)], which, in view of the monotonicity of the L^p-norm in the preorder $\prec$ [**2**, Theorem (2.5)], implies $\|f_\uparrow - g_\uparrow\|_p \le \|f - g\|_p$. Note also that for $p = 2$ the nonexpansive property of the operator $\uparrow$ follows directly from the Hardy-Littlewood-Pólya inequality $\langle f, g\rangle_\mu \le \langle f_\uparrow, g_\uparrow\rangle$.

For $p = +\infty$, the rearrangement operator is nonexpansive in the essential supremum norm. For an economic application in [**6**], however, it is necessary to establish the continuity of the rearrangement operator in some topologies on $L^\infty(\mu)$ and L^∞ in which the respective continuous duals are $L^1(\mu)$ and L^1. From the following example it is seen that the rearrangment operator on L^∞ into L^∞ is discontinuous in the weak topology $\sigma(L^\infty, L^1)$.

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(2.1) EXAMPLE. Let $f^n(t) = \operatorname{sgn} \sin 2\pi nt$ for $t \in [0,1]$ and $n = 1, 2, \ldots$. One has $f^n \to 0$ in $\sigma(L^\infty, L^1)$; but for all n, f^n is equal to (-1) on $[0, 1/2)$ and to $+1$ on $[1/2, 1]$.

As this example suggests, continuity holds for the Mackey topology.

(2.2) THEOREM. *The rearrangement operator $\uparrow$ is a uniformly continuous mapping of $L^\infty(\mu)$ endowed with the Mackey topology $\tau(L^\infty(\mu), L^1(\mu))$ into L^∞ endowed with the Mackey topology $\tau(L^\infty, L^1)$.*

PROOF. The Mackey topology $\tau(E, F)$ is the topology of uniform convergence on $\sigma(F, E)$-compact sets (see, for example, [**9**, Corollary 1 to Theorem IV.3.2, and subsequent remarks]). By the Dunford-Pettis compactness criterion, weak relative compactness is equivalent to uniform integrability (see [**5**, Theorems IV.8.9 and V.6.1]).

Take any uniformly integrable $\Phi \subset L^1$. To prove the Theorem, it is necessary and sufficient to find a uniformly integrable $\Psi \subset L^1(\mu)$ such that, for any $f, g \in L^\infty(\mu)$, if $|\langle \psi, g - f\rangle_\mu| < 1$ for all $\psi \in \Psi$, then $|\langle \phi, g_\uparrow - f_\uparrow\rangle| < 1$ for all $\phi \in \Phi$.

From [**3**, (6.1)(ii)],

$$g_\uparrow - f_\uparrow \prec g - f \quad \text{and} \quad f_\uparrow - g_\uparrow \prec f - g. \tag{2.3}$$

By the Hardy-Littlewood-Pólya inequality (see, for example, [**4**, Lemma (3.4)]), (2.3) implies

$$\langle \phi, g_\uparrow - f_\uparrow\rangle \le \langle \phi_\uparrow, (g - f)_\uparrow\rangle \quad \text{for all } \phi \in L^1 \tag{2.4}$$

and

$$\langle \phi, f_\uparrow - g_\uparrow\rangle \le \langle \phi_\uparrow, (f - g)_\uparrow\rangle \quad \text{for all } \phi \in L^1.$$

This last inequality can also be written equivalently as

$$\langle \phi, g_\uparrow - f_\uparrow\rangle \ge \langle \phi_\downarrow, (g - f)_\uparrow\rangle \quad \text{for all } \phi \in L^1. \tag{2.5}$$

Consider first the case of a nonatomic μ. Let $\mathcal{S}$ be the set of all measure-preserving mappings of $(A, \mathcal{A}, \mu)$ into $[0,1]$. By [**4**, Proposition (3.3) or **8**, Lemma 1], for any f and g there exists $S \in \mathcal{S}$ such that $g - f = (g - f)_\uparrow \circ S$. Then for such S,

$$\langle h, (g - f)_\uparrow\rangle = \langle h \circ S, g - f\rangle_\mu \quad \text{for all } h \in L^1. \tag{2.6}$$

Using (2.6) for $h = \phi_\uparrow, \phi_\downarrow$ with (2.4) and (2.5) gives

$$\inf_{S \in \mathcal{S}} \langle \phi_\downarrow \circ S, g - f\rangle_\mu \le \langle \phi, g_\uparrow - f_\uparrow\rangle \le \sup_{S \in \mathcal{S}} \langle \phi_\uparrow \circ S, g - f\rangle_\mu \quad \text{for all } \phi \in L^1. \tag{2.7}$$

Consider the set $\Phi_\uparrow \circ \mathcal{S} = \{\phi_\uparrow \circ S \mid \phi \in \Phi, S \in \mathcal{S}\}$: It is a uniformly integrable subset of $L^1(\mu)$ because uniform integrability is a property which depends only on the distributions of the functions involved. It follows from (2.7) that it suffices to take $\Psi = \Phi_\uparrow \circ \mathcal{S}$. This completes the proof for a nonatomic μ.

Only a slight modification of the preceding argument has to be made for a μ with atoms. By [**7**, Theorem (7.1) or **3**, p. 390, first line], there exist $\bar{\phi}$, $\underline{\phi} \in L^1(\mu)$ such that

$$\langle \phi_\uparrow, (g - f)_\uparrow\rangle = \langle \bar{\phi}, g - f\rangle_\mu \quad \text{and} \quad \bar{\phi} \prec \phi, \tag{2.8}$$

and

$$\langle \phi_{\downarrow}, (g-f)_{\uparrow} \rangle = \langle \underline{\phi}, g-f \rangle_{\mu} \quad \text{and} \quad \underline{\phi} \prec \phi. \tag{2.9}$$

Let $\Omega(\phi) = \{\psi \in L^1(\mu) | \psi \prec \phi\}$ and $\Omega(\Phi) = \bigcup_{\phi \in \Phi} \Omega(\phi)$. By [**1**, Corollary 4.3 and [**7**, Theorem 9.5, the set $\Omega(\Phi)$ is uniformly integrable because Φ is. From (2.4), (2.5), (2.8) and (2.9) it follows that

$$\begin{aligned} \inf_{\psi \in \Omega(\Phi)} \langle \psi, g-f \rangle_{\mu} &\leq \langle \phi, g_{\uparrow} - f_{\uparrow} \rangle \\ &\leq \sup_{\psi \in \Omega(\Phi)} \langle \psi, g-f \rangle_{\mu} \quad \text{for all } \phi \in \Phi. \end{aligned} \tag{2.10}$$

It follows from (2.10) that it suffices to take $\Psi = \Omega(\Phi)$. Q.E.D.

(2.11) REMARK. For countable sequences, convergence in the Mackey topology $\tau(L^\infty(\mu), L^1(\mu))$ is equivalent to convergence in the measure μ and boundedness in the essential supremum norm. Therefore, *sequential* continuity of the rearrangement operator in the Mackey topologies under consideration follows also from its continuity in the topologies of convergence in the measures μ and mes.

(2.12) REMARK. Even after their restriction to the range of the operator $\uparrow$, viz., the cone of nondecreasing functions in L^∞, the Mackey topology $\tau(L^\infty, L^1)$ is strictly stronger than the weak topology $\sigma(L^\infty, L^1)$.

REFERENCES

1. K. M. Chong, *Spectral orders, uniform integrability and Lebesgue's dominated convergence theorem*, Trans. Amer. Math. Soc. **191** (1974), 395–404.
2. ——, *Some extensions of a theorem of Hardy, Littlewood and Pólya and their applications*, Canad. J. Math. **26** (1974), 1321–1340.
3. P. W. Day, *Rearrangement inequalities*, Canad. J. Math. **24** (1972), 930–943.
4. ——, *Decreasing rearrangements and doubly stochastic operators*, Trans. Amer. Math. Soc. **178** (1973), 383–392.
5. N. Dunford and J. T. Schwartz, *Linear operators. Part* I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958.
6. A. Horsley and A. Wrobel, *The formal theory of pricing and investment for electricity.* I: *A continuous-time model of deterministic production*, ICERD Discussion Paper No. 86/135, London School of Economics.
7. W. A. J. Luxemburg, *Rearrangement invariant Banach function spaces*, Queen's Papers in Pure and Appl. Math. **10** (1967), 83–144.
8. J. V. Ryff, *Orbits of L^1-functions under doubly stochastic transformations*, Trans. Amer. Math. Soc. **117** (1965), 92–100.
9. H. H. Schaefer, *Topological vector spaces*, Graduate Texts in Math., vol. 3, Springer-Verlag, Berlin and New York, 1971.

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