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DENTING POINTS IN BOCHNER L^p-SPACES[1]

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ABSTRACT. A characterization of denting points in the unit balls of $L^p(\mu, X)$, $1 < p < \infty$, is given. This characterization is compared to similar known results concerning strongly extreme points and extreme points.

It is known [**Su**] that if $f \in L^p(\mu, X)$, $1 < p < \infty$, satisfies that $\|f\| = 1$ and, for almost all t in the support of f, the element $f(t)/\|f(t)\|$ is an extreme point in the unit ball of X, then f is an extreme point in the unit ball of $L^p(\mu, X)$. The converse was established, with some restrictions on the measure space, first for separable dual spaces [**Su**] and then for all separable spaces [**J1**]; just recently [**G4**] the restrictions on measure space were removed. However, there exists [**G3**] a nonseparable Banach space X for which the converse does not hold. For the strongly extreme and strongly exposed points, the natural conditions on the values of f are known to be necessary [**G2**] for f to be such an extreme point in the unit ball of $L^p(\mu, X)$. These natural conditions are also sufficient for strongly extreme points [**S1**] and sufficient for strongly exposed points [**G1**] whenever X is smooth. We refer to [**S2**] for a summary of these and related results.

Recall that an element x in X is a denting point of the unit ball if $\|x\| = 1$ and $x \notin \overline{\text{co}}(M(x, \varepsilon))$ for all $\varepsilon > 0$ where $M(x, \varepsilon) = \{y \in X \colon \|y\| \leq 1 \text{ and } \|x - y\| > \varepsilon\}$. A Banach space X is said to have property (G) if every point of the unit sphere is a denting point of the unit ball of X. For the relationships between property (G) and other geometric properties of X, see [**F-G**]. In this paper, we continue our study on denting points [**L-L**]. The main result of the paper is the following theorem.

THEOREM 1. *Let X be any Banach space. f is a denting point in the unit ball of $L^p(\mu, X)$, $1 < p < \infty$, if and only if $\|f\| = 1$ and, for almost all t in the support of f, the element $f(t)/\|f(t)\|$ is a denting point of the unit ball of X.*

PROOF. Assume that f, $\|f\| = 1$, is not a denting point in the unit ball of $L^p(\mu, X)$. Then there exist $\varepsilon > 0$ and sequences $(a_i^m)_{i=1}^{N_m}$ and (f_n) such that $\|f_n\| \leq 1$, $\|f - f_n\| > \varepsilon$, $a_i^m \geq 0$, $\sum_{i=1}^{N_m} a_i^m = 1$, and $\|\sum_{i=1}^{N_m} a_i^m f_i - f\| < 1/m$. Since $\sum_{i=1}^{N_m} a_i^m \|f_i(\cdot)\| \geq \|\sum_{i=1}^{N_m} a_i^m f_i(\cdot)\|$ and

$$1 \geq \sum_{i=1}^{N_m} a_i^m \|f_i\| \geq \left\| \sum_{i=1}^{N_m} a_i^m \|f_i(\cdot)\| \right\|_{L^p(\mu)},$$

$$\lim_{m \to \infty} \left(\sum_{i=1}^{N_m} a_i^m \|f_i(\cdot)\| - \left\| \sum_{i=1}^{N_m} a_i^m f_i(\cdot) \right\| \right) = 0 \quad \text{in } L^p(\mu).$$

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Otherwise, by uniform convexity of $L^p(\mu)$, there exists $\delta > 0$ such that

$$\left\| \left\| \sum_{i=1}^{N_m} a_i^m f_i(\cdot) \right\| \right\|_{L^p(\mu)} \leq \frac{1}{2} \left\| \sum_{i=1}^{N_m} a_i^m \|f_i(\cdot)\| + \left\| \sum_{i=1}^{N_m} a_i^m f_i(\cdot) \right\| \right\|_{L^p(\mu)} \leq 1 - \delta$$

for infinitely many m. This contradicts the fact that

$$\lim_{m\to\infty} \left\| \left\| \sum_{i=1}^{N_m} a_i^m f_i(\cdot) \right\| \right\|_{L^p} = \|f\| = 1.$$

So

$$\lim_{m\to\infty} \sum_{i=1}^{N_m} a_i^m \|f_i(\cdot)\| = \|f(\cdot)\| \quad \text{in } L^p(\mu).$$

Let

$$M_{m,\varepsilon} = \{i\colon a_i^m \neq 0 \text{ and } \| \|f_i(\cdot)\| - \|f(\cdot)\| \|_{L^p(\mu)} > \varepsilon\}.$$

Since $L^p(\mu)$ is uniformly convex (so every element with norm 1 is a denting point of the unit ball), one can verify that

$$\lim_{m\to\infty} \sum_{i\in M_{m,\varepsilon}} a_i^m = 0.$$

Let

$$\bar{f}_n(t) = \begin{cases} f_n(t) & \text{if } \|f_n(t)\| \leq \|f(t)\|, \\ \|f(t)\| f_n(t)/\|f_n(t)\| & \text{otherwise.} \end{cases}$$

Clearly, $\|f_n(\cdot) - \bar{f}_n(\cdot)\| \leq | \|f_n(\cdot)\| - \|f(\cdot)\| |$. So

$$\begin{aligned} &\overline{\lim_{m\to\infty}} \left\| \sum_{i=1}^{N_m} a_i^m \|f_i(\cdot) - \bar{f}_i(\cdot)\| \right\|_{L^p(\mu)} \\ &\quad \leq \overline{\lim_{m\to\infty}} \left(\left\| \sum_{i\in M_{m,\varepsilon}} a_i^m \|f_i(\cdot) - \bar{f}_i(\cdot)\| \right\|_{L^p(\mu)} + \left\| \sum_{i\notin M_{m,\varepsilon}} a_i^m \|f_i(\cdot) - \bar{f}_i(\cdot)\| \right\|_{L^p(\mu)} \right) \\ &\quad \leq \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have $\lim_{m\to\infty} \sum_{i=1}^{N_m} a_i^m \|f_i(\cdot) - \bar{f}_i(\cdot)\| = 0$ and

$$\lim_{m\to\infty} \sum_{i=1}^{N_m} a_i^m \bar{f}_i = f.$$

If $\|f_i - \bar{f}_i\| < \varepsilon/2$, then $\|\bar{f}_i - f\| \geq \|f_i - f\| - \|f_i - \bar{f}_i\| \geq \varepsilon/2$. On the other hand, if $\|f_i - \bar{f}_i\| \geq \varepsilon/2$, then

$$\|\bar{f}_i\|^p \leq \|f_i\|^p - (\varepsilon/2)^p \leq 1 - (\varepsilon/2)^p = (1-\delta)^p$$

(because $\|f_i(\cdot)\| = \|\bar{f}_i(\cdot)\| + (\|f_i(\cdot)\| - \|\bar{f}_i(\cdot)\|)$) and $\|f - \bar{f}_i\| \geq \delta$. So $\|f - \bar{f}_i\| \geq \min(\varepsilon/2, \delta) > 0$. The proof of Theorem 2 in **[L-L]** shows that there is a subset B of the $\operatorname{supp} f$ with positive measure such that $f(t)/\|f(t)\|$ is not a denting point in the unit ball of X for almost all $t \in B$. To prove the converse direction, let $A_{n,m,l}$ denote the set

$$\left\{x \in X\colon \|x\| = 1 \text{ and there exist } x_1, x_2, \ldots, x_{2^n} \in M\left(x, \frac{1}{l}\right) \right.$$
$$\left. \text{such that } \left\| \frac{1}{2^n} \sum_{i=1}^{2^n} x_i - x \right\| < \frac{1}{m} \right\}.$$

If x, $\|x\| = 1$, is not a denting point of the unit ball, then $\exists l \,\forall m \,\exists n$ such that $x \in A_{n,m,l}$. Clearly, $A_{n,m,l}$ is relatively open in the unit sphere of X, and $A_{n,m+1,l} \subseteq A_{n,m,l} \subseteq A_{n+1,m,l} \subseteq A_{n+1,m,l+1}$. Let f be an element in $L^p(\mu, X)$ such that $\|f\| = 1$ and

$$\{t\colon t \in \operatorname{supp} f \text{ and } f(t)/\|f(t)\| \text{ is not a denting point of the unit ball of } X\}$$
$$= \left\{ t \in \operatorname{supp} f\colon f(t)/\|f(t)\| \in \bigcup_{l=1} \bigcap_{m=1} \bigcup_{n=1} A_{n,m,l} \right\}$$

has positive measure. Hence, there exists L such that

$$\left\{t \in \operatorname{supp} f\colon f(t)/\|f(t)\| \in \bigcup_{l=1}^{L} \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{n,m,l} \right\}$$
$$= \left\{ t \in \operatorname{supp} f\colon f(t)/\|f(t)\| \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{n,m,L} \right\} = B$$

has positive measure. Let

$$B_{n,m} = \{t \in \operatorname{supp} f\colon f(t)/\|f(t)\| \in A_{n,m,L}\}.$$

Then $B = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} B_{n,m}$. Since $B \subseteq \bigcup_{n=1}^{\infty} B_{n,m} \subseteq \operatorname{supp} f$ and $\mu(B) > 0$ for each m_0 there exists n_0 such that

$$\int_{B_{n_0,m_0}} \|f(\cdot)\|^p \, d\mu \geq \frac{1}{2} \int_B \|f(\cdot)\|^p \, d\mu = \alpha > 0.$$

(Note: $B_{n,m} \subseteq B_{n+1,m}$.) Given $M > 0$, let $m_0 = \max(10M, 10L)$. Let $B(x, \varepsilon)$ denote the set

$$B(x, \varepsilon) = \{y\colon \|y - x\| < \varepsilon\}.$$

If $y_0 \in B(x, 1/m_0) \cap A_{n_0,m_0} \neq \emptyset$, there exist $z_1, z_2, \ldots, z_{2^{n_0}} \in M(y_0, 1/L)$ such that

$$\|y_0 - z_i\| > 1/L \qquad \forall i = 1, 2, \ldots, 2^{n_0}$$

and

$$\left\| y_0 - \frac{1}{2^{n_0}} \sum_{i=1}^{2^{n_0}} z_i \right\| < \frac{1}{m_0}.$$

Hence, if $y \in B(x, 1/m_0)$, then $\|y - z_i\| > 1/L - 2/m_0 \geq 8/10L$ and

$$\left\| y - \frac{1}{2^{n_0}} \sum_{i=1}^{2^{n_0}} z_i \right\| < \frac{1}{m_0} + \frac{2}{m_0} = \frac{3}{m_0} < \frac{1}{M}.$$

Since $f \in L^p(\mu, X)$, we may assume that the range of f is separable. Let $\{y_1, y_2, \ldots, y_n, \ldots\}$ be a dense subset of the range. If $B(y_i/\|y_i\|, 1/m_0) \cap A_{n_0,m_0,L} \neq \emptyset$, then let $z_{i,1}, z_{i,2}, \ldots, z_{i,2^{n_0}}$ be 2^{n_0} elements in the unit ball such that if $y \in B(y_i/\|y_i\|, 1/m_0)$ then $\|y - z_{i,j}\| \geq 8/10L$ for $j = 1, 2, \ldots, 2^{n_0}$ and

$$\left\| y - \frac{1}{2^{n_0}} \sum_{j=1}^{2^{n_0}} z_{i,j} \right\| < \frac{1}{M}.$$

For fixed $t \in \operatorname{supp} f$, let i be the first k such that $f(t)/\|f(t)\| \in B(y_k/\|y_k\|, 1/m_0)$. Then let

$$f_j(t) = \begin{cases} f(t) & \text{if } t \notin \operatorname{supp} f \text{ or } B(y_i/\|y_i\|, 1/m_0) \cap A_{n_0,m_0,L} = \emptyset, \\ \|f(t)\| z_{i,j} & \text{otherwise.} \end{cases}$$

Clearly f_j, $j = 1, 2, \ldots, 2^{n_0}$, is measurable and $\|f_j - f\| > 8\alpha/10L$, and

$$\left\| \frac{1}{2^{n_0}} \sum f_j - f \right\| < \frac{1}{M}.$$

So f is not a denting point of the unit ball of $L^p(\mu, X)$.

Recall that an element x of the unit sphere of a Banach space X is a strongly extreme point of the unit ball if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the conditions $\|x \pm z\| < 1 + \delta$ imply $\|z\| < \varepsilon$. In other words, if x is not a strongly extreme point of the unit ball then there exists $\varepsilon > 0$ such that x is in the closure of $\{\frac{1}{2}(y_1 + y_2) \colon y_1, y_2 \in M(x, \varepsilon)\}$. We have the following corollary to the proof of Theorem 1.

COROLLARY 2 [**G2**]. *If f is a strongly extreme point of the unit ball of $L^p(\mu, X)$, then for almost all t in the support of f, the element $f(t)/\|f(t)\|$ is a strongly extreme point of the unit ball of X.*

The following theorem is a strengthening of Theorem 2 in [**J2**] when the set T is the strongly exposed points of the unit ball of X.

THEOREM 3. *Let T be a set of norm one elements in X and let S denote the set of all functions in $L^p(\mu, X)$, $1 < p < \infty$, such that $f = \sum_{j=1}^n x_j X_{A_j}$, $\|f\| = 1$ and $x_j/\|x_j\|$ is in T. If the unit ball of X is the closed convex hull of T, then the unit ball of $L^p(\mu, X)$ is the closed convex hull of S.*

PROOF. Let g be an element of the unit ball of $L^p(\mu, X)$ and let $\varepsilon > 0$ be given. Without loss of generality assume $\|g\| = 1$ and g is a simple function, say $g = \sum_{i=1}^m w_i X_{A_i}$. By hypothesis, each $w_i/\|w_i\|$ is in $\overline{\text{co}}(T)$ for $i = 1, \ldots, m$ and hence there exist n and n elements $z_1, \ldots, z_n$ in T such that, for each $i = 1, \ldots, m$, elements $y_j^{(i)}$ for $j = 1, \ldots, n$ may be chosen from $\{z_1, \ldots, z_n\}$ such that

$$(*) \qquad \left\| \frac{1}{n} \sum_{j=1}^{n} y_j^{(i)} - \frac{w_i}{\|w_i\|} \right\| < \varepsilon.$$

For each $j = 1, \ldots, n$, let $f_j = \sum_{i=1}^{m} \|w_i\| y_j^{(i)} X_{A_i}$ and note that f_j is an element of S. By computing, it follows that

$$\begin{aligned}
\left\| \frac{1}{n} \sum_{j=1}^{n} f_j - g \right\|^p &= \left\| \sum_{i=1}^{m} \left(\|w_i\| \frac{1}{n} \sum_{j=1}^{n} y_j^{(i)} - w_i \right) X_{A_i} \right\|^p \\
&= \sum_{i=1}^{m} \int_{A_i} \left\| \left(\|w_i\| \frac{1}{n} \sum_{j=1}^{n} y_j^{(i)} - w_i \right) \right\|^p d\mu \\
&< \sum_{i=1}^{m} \int_{A_i} \varepsilon^p \|w_i\|^p \, d\mu \quad \text{by } (*) \\
&= \varepsilon^p \quad \text{since } \|g\| = 1
\end{aligned}$$

and hence g is in $\overline{\text{co}}(S)$. This completes the proof.

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REFERENCES

[**F-G**] K. Fan and I. Glicksberg, *Some geometric properties of the spheres in normed linear space*, Duke Math. J. **25** (1958), 553–568.

[**G1**] P. Greim, *Strongly exposed points in Bochner L^p-spaces*, Proc. Amer. Math. Soc. **88** (1983), 81–84.

[**G2**] ——, *A note on strong extreme and strongly exposed points in Bochern L^p-spaces*, Proc. Amer. Math. Soc. **93** (1985), 65–66.

[**G3**] ——, *An extremal vector-valued L^p-function taking no extremal vectors as values*, Proc. Amer. Math. Soc. **84** (1982), 65–68.

[**G4**] ——, *An extension of J. A. Johnson's characterization of extremal vector-valued L^p-functions*, preprint.

[**J1**] J. A. Johnson, *Extreme measurable selections*, Proc. Amer. Math. Soc. **44** (1974), 107–111.

[**J2**] ——, *Strongly exposed points in $L^p(\mu, E)$*, Rocky Mountain J. Math. **10** (1980), 517–519.

[**L-L**] Bor-Luh Lin and Pei-kee Lin, *Property* (H) *in Lebesgue-Bochner function spaces*, Proc. Amer. Math. Soc. **95** (1985), 581–584.

[**S1**] M. A. Smith, *Strongly extreme points in $L^p(\mu, X)$*, Rocky Mountain J. Math. (to appear).

[**S2**] ——, *Rotundity and extremity in $l^p(X_i)$ and $L^p(\mu, X)$*, preprint.

[**Su**] K. Sundaresan, *Extreme points of the unit cell in Lebesgue-Bochner function spaces*, Colloq. Math. **22** (1970), 111–119.

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