FINITE RANK PERTURBATIONS OF SINGULAR SPECTRA

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ABSTRACT. Let $T$ be selfadjoint, and $V$ nonnegative of finite rank, with the range of $V$ cyclic for $T$. Then the singular parts of $T$ and $H = T + V$ are supported on two sets $S_T$ and $S_H$ such that the multiplicity of $T$ on $S_T \cap S_H$ is less than the rank of $V$.

1. Introduction. In [3], Donoghue, following earlier work of Aronszajn, proved

1. THEOREM. Let $\phi$ be a cyclic vector for a selfadjoint operator $T$. For real $c \neq 0$, the singular parts of $T$ and $H = T + c\langle \cdot, \phi \rangle \phi$ are supported on disjoint sets.

To generalize this result to perturbations of rank higher than one is not completely straightforward, as a consideration of matrix examples will easily show. A certain generalization to nonnegative perturbations was given by the author in [4]. The criterion of that paper will be applied here to prove the following result:

2. THEOREM. Let $T$ be selfadjoint, $V$ a nonnegative operator of finite rank, and $H = T + V$. Assume that the range of $V$ is cyclic for $T$. Let $\mu_T$ and $n_T(\lambda)$ be a scalar spectral measure and multiplicity function of $T$, and define

$$G = \{ \lambda : n_T(\lambda) = \text{rank} V \}.$$ 

Then there exist sets $S_T$ and $S_H$ supporting the singular parts of $T$ and $H$ such that $S_T \cap G$ and $S_H$ are disjoint.

Note that $n_T(\lambda)$ cannot exceed the rank of $V$ when the range of $V$ is cyclic. This has the following corollary, which is interesting even for matrices.

3. COROLLARY. Let $T$ be selfadjoint, $V$ nonnegative of finite rank, and the range of $V$ cyclic for $T$. If $\lambda$ is an eigenvalue of $T$ with multiplicity equal to the rank of $V$, then $\lambda$ is not an eigenvalue of $H = T + V$.

For related work, see also [1, 2, 5].

2. Proofs. Note that the basic Hilbert space $\mathcal{H}$ is separable because the range $V \mathcal{H}$ of $V$ is cyclic. Multiplicity theory is therefore applicable.

For $\varepsilon > 0$, define the function

$$\delta_\varepsilon(t) = \frac{\varepsilon}{\pi t^2 + \varepsilon^2}.$$ 

As observed by Donoghue [3, §1], the singular part of $T$ is supported by the measurable set

$$S_T = \left\{ \lambda : \lim_{\varepsilon \downarrow 0} \langle \delta_\varepsilon (T - \lambda) x, x \rangle = \infty \text{ for some } x \in V \mathcal{H} \right\}$$ 

and similarly for $H$. 

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Let \( v_1 \geq v_2 \geq \cdots \geq v_r > 0 \) be the nonzero eigenvalues of \( V \), and \( \phi_1, \ldots, \phi_r \) the corresponding normalized eigenvectors, which therefore are an orthonormal basis of \( V \mathcal{H} \). Because \( V \mathcal{H} \) is cyclic, one may choose
\[
\mu_T(S) = \sum_{j=1}^{r} \langle E_T[S] \phi_j, \phi_j \rangle
\]
as the scalar spectral measure, where \( E_T \) is the spectral measure of \( T \). Thus, \( \langle E_T(d\lambda) \phi_i, \phi_j \rangle \) is absolutely continuous with respect to \( \mu_T \). Define \( M_{ij}(\lambda) \) \( \mu_T \)-a.e. to be the Radon-Nikodým derivative
\[
\langle E_T(d\lambda) \phi_i, \phi_j \rangle = M_{ij}(\lambda) \mu_T(d\lambda)
\]
and \( M(\lambda) \) to be the nonnegative matrix \( M(\lambda) = \{M_{ij}(\lambda)\}_{i,j=1,\ldots,r} \). We shall regard \( M(\lambda) \) as an operator on the space \( V \mathcal{H} \). Let \( m(\lambda) \) be the smallest eigenvalue of \( M(\lambda) \):
\[
m(\lambda) = \inf\{\langle M(\lambda)u, u \rangle : |u|^2 = 1, \ u \in V \mathcal{H} \},
\]
where \( |u| \) is the norm of \( u \) in \( \mathcal{H} \). Since \( u \) may be restricted to a countable dense set, \( m(\lambda) \) is measurable. One has
\[
(1) \quad M(\lambda) \geq m(\lambda) P
\]
where \( P \) is the projection onto \( V \mathcal{H} \). Note also that
\[
(2) \quad m(\lambda) \leq 1.
\]
Clearly, for all Borel sets \( S \), \( \langle E_T[S] \phi_i, \phi_i \rangle \leq \mu_T(S) \) so that \( M_{ii}(\lambda) \leq 1 \) \( \mu_T \)-a.e., and hence
\[
m(\lambda) \leq \min_{1 \leq i \leq r} M_{ii}(\lambda) \leq 1.
\]

4. LEMMA. One has
\[
n_T(\lambda) = \text{rank } M(\lambda) \quad \mu_T \text{-a.e.}
\]

PROOF. This undoubtedly follows from the readers' favorite version of multiplicity theory. The author's favorite version is the Kato-Kuroda construction of direct integrals by spectral forms \[6, 7\]. In that terminology, let \( X = V \mathcal{H} \), and
\[
f(\lambda, u) = \sum_{i,j=1}^{r} u_i \overline{u}_j M_{ij}(\lambda) \equiv \langle M(\lambda)u, u \rangle
\]
for \( u = u_1 \phi_1 + \cdots + u_r \phi_r \in X \). Then \( (f, X) \) is a spectral form for \( T \) with respect to \( \mu_T \), and the direct integral
\[
\mathcal{H} \cong \int_{\sigma(T)}^{\oplus} X(\lambda) \mu_T(d\lambda)
\]
diagonalizes \( T \), where \( X(\lambda) \) is the (completion of) the quotient space \( X/\{u \in X : f(\lambda, u) = 0\} \). In this case, no completion is needed, since \( X(\lambda) \) is the finite-dimensional space \( V \mathcal{H}/\ker M(\lambda) \) whose dimension is \( \text{rank } M(\lambda) \). \( \square \)

The theorem of \[4\] will now be recalled. Let \( K \) be another Hilbert space, and \( A : K \to \mathcal{H} \) bounded. Let \( T \) be selfadjoint, and assume that \( AK \) is cyclic for \( T \).
5. PROPOSITION. The singular part of \( H = T + AA^* \) is supported on the complement of the set of points \( \lambda \) for which there is an \( \eta > 0 \) such that

\[
A^*\delta_{\epsilon}(T - \lambda)A \geq \eta I
\]

for all sufficiently small \( \epsilon > 0 \).

Note that \( I \) in (3) is the identity on \( K \), not \( \mathcal{H} \). Note also that Proposition 5 implies Theorem 1 if one takes \( K = \mathbb{C} \) (the complex numbers) and \( A^* = c^{1/2}\langle \cdot, \phi \rangle \) (not \( A \), as the misprint in [4] has it). Note finally that although \( T \) was assumed bounded in [4], the proof there goes through unchanged for unbounded \( T \).

To prove Theorem 2, factor \( V = AA^* \) through the space \( K = V\mathcal{H} = V^{1/2}\mathcal{H} \) by defining \( A: V^{1/2}\mathcal{H} \to \mathcal{H} \) as \( Au = V^{1/2}u \). Then \( A^*: \mathcal{H} \to V^{1/2}\mathcal{H} \) is also \( A^*u = V^{1/2}u \), and \( V = AA^* \). The identity \( I \) in (3) is now the projection \( P \) onto \( V\mathcal{H} \).

For fixed \( \lambda \) and \( u \in V\mathcal{H} \), one has

\[
(A^*\delta_{\epsilon}(T - \lambda)Au, u) = (\delta_{\epsilon}(T - \lambda)V^{1/2}u, V^{1/2}u)
\]

\[
= \sum_{i,j=1}^{r} u_i\bar{u}_j v_i^{1/2} v_j^{1/2} (\delta_{\epsilon}(T - \lambda)\phi_i, \phi_j)
\]

\[
\geq v_r(\delta_{\epsilon}(T - \lambda)u, u) = v_r \int \delta_{\epsilon}(t - \lambda)(M(t)u, u) \mu_T(dt)
\]

\[
\geq v_r\|u\|^2 \int \delta_{\epsilon}(t - \lambda)m(t) \mu_T(dt)
\]

where (1) was used at the last step. Let \( F \) be the set of all \( \lambda \) for which

\[
\lim_{\epsilon \to 0} \int \delta_{\epsilon}(t - \lambda)m(t) \mu_T(dt) = \infty.
\]

By (2), \( F \subset S_T \), while by Proposition 5, its complement \( F^c \) supports the singular part of \( H \). (In fact, by the proof in [5], \( S_H \subset F^c \).)

Now [3, §1] \( F \) supports the singular part of the measure \( m(t) \mu_T(dt) \). The measure \( \chi_G(t) \mu_T(dt) \) has the same null sets, because \( G = \{ t: m(t) > 0 \} \), and its singular part is supported by \( S_T \cap G \). Thus \( F \) and \( S_T \cap G \) differ only by a set of \( \mu_T \)-measure zero. The result is then obtained by replacing \( S_T \) by \( S'_T = S_T - (G \cap S_T \cap F^c) \). □

REFERENCES


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