PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 4, August 1986

A COUNTEREXAMPLE TO A RESULT CONCERNING CONTROLLED APPROXIMATION[1]

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ABSTRACT. A result of Strang and Fix states that if the order of controlled approximation from a collection of locally supported elements is k, then there is a linear combination Ω of those elements and their translates such that any polynomial of degree less than k can be reproduced by Ω and its translates. This paper gives a counterexample to their result.

This paper concerns the so-called controlled approximation. This concept was introduced by Strang [**S**] in 1970. In that paper, Strang described a systematic approach to the choice of trial functions used in finite element analysis. (Also see [**FS**] for some related results.) To describe his approach, we first introduce some notation. As usual, we mean by $W^{k,p}(\mathbf{R}^n)$ the usual Sobolev space with norm

$$\|u\|_{k,p} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p}.$$

(Here we use standard multi-index notation.) The seminorm $|\cdot|_{k,p}$ on $W^{k,p}(\mathbf{R}^n)$ is defined to be

$$|u|_{k,p} := \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p}.$$

When $k=0$, $|\cdot|_{k,p}$ is a norm, and we write $\|u\|_p := |u|_{0,p} = \|u\|_{L^p}$. By $W_c^{k,p}(\mathbf{R}^n)$ we denote the space of all functions in $W^{k,p}(\mathbf{R}^n)$ that have compact support. By $l^p(\mathbf{Z}^n)$ we mean the space of the mappings w from $\mathbf{Z}^n$ to $\mathbf{R}$ which satisfy

$$\|w\|_p := \left(\sum_j |w(j)|^p\right)^{1/p} < \infty.$$

Strang's approach can be described as follows: Choose one or more functions $\phi_1,\dots,\phi_N$ in $W_c^{k,2}(R^n)$, rescale the independent variables, obtaining $h^{-n/2}\phi_i(\cdot/h)$ with $h>0$ ($i=1,\dots,N$), and translate the functions just constructed, replacing x by $x-jh$. Thus we have a family of trial functions:

$$\phi_{i,j}^h(x) := h^{-n/2}\phi_i(x/h - j), \qquad i=1,\dots,N,\ j\in\mathbf{Z}^n.$$

When $h=1$, we omit the superscript h. Note that the coefficient $h^{-n/2}$ normalizes $\phi_{i,j}^h$ in the sense that $\|\phi_{i,j}^h\|_2 = \|\phi_i\|_2$.

A principal question in finite element analysis is the degree of approximation which can be achieved by the span of $\phi_{i,j}^h$. Concerning this problem, Strang and Fix state the following result (see [**SF**, Theorem II]):

Received by the editors September 24, 1984 and, in revised form July 29, 1985.
1980 *Mathematics Subject Classification*. Primary 41A29.

[1]Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-8210950.

THEOREM A. *Suppose $\phi_1, \ldots, \phi_N$ are in $W_c^{k,2}(\mathbf{R}^n)$. Then the following conditions are equivalent:*

$(\mathrm{i})_k$ *For each u in $W^{k+1,2}$ and $h > 0$, there are weights $w_{i,j}^h$ such that*

$$\left| u - \sum_{i,j} w_{i,j}^h \phi_{i,j}^h \right|_{s,2} \leq \mathrm{const}_s h^{k+1-s} |u|_{k+1,2} \quad \text{for } s \leq k, \tag{1}$$

$$\|w_i^h\|_2 \leq \mathrm{const}\|u\|_2. \tag{2}$$

(*Here w_i^h is the mapping $j \to w_{i,j}^h$, $j \in \mathbf{Z}^n$.*)

$(\mathrm{ii})_k$ *There is a finite linear combination Ω of $\phi_1, \ldots, \phi_N$ and their translates $\phi_{i,j}$ such that*

for $|\alpha| \leq k$, $\sum_{j \in \mathbf{Z}^n} j^\alpha \Omega(x - j)$ is a polynomial in $x_1, \ldots, x_n$ with leading term x^α.

If a collection $\Phi = \{\phi_1, \ldots, \phi_N\}$ satisfies $(\mathrm{i})_{k-1}$ but not $(\mathrm{i})_k$, then we say that Φ has controlled approximation order k. Note that (2) constrains the coefficients $w_{i,j}^h$. This is the reason why [**S**] calls such approximation *controlled.*

The implication of $(\mathrm{i})_k$ from $(\mathrm{ii})_k$ is in no doubt. This has been proved by several authors, in particular, by [**SF**] (see the references cited by [**S**] and [$\mathbf{FS}_2$]). But, the other direction, i.e., the implication of $(\mathrm{ii})_k$ from $(\mathrm{i})_k$ has been proved *only* for $N = 1$ by [**SF**]; the proof for the case $N > 1$ is not discussed in [**SF**].

Recently, the rapid development of the theory of multivariate spline functions has given a new impetus to the controlled approximation problem. In [**DM**], Theorem A was cited to determine the controlled approximation order from certain bivariate box splines. In order to advance the theory of controlled approximation, it becomes important to answer the question whether $(\mathrm{i})_k$ implies $(\mathrm{ii})_k$.

As it turns out, $(\mathrm{i})_k$ does *not* imply $(\mathrm{ii})_k$ in general. We shall give a counterexample in the following. In our example, $n = 2$, $N = 4$, and $\phi_1, \ldots, \phi_4$ are splines with compact support, which are constructed from box splines. Thus we have to recall box splines. Before doing so, we need to introduce more notation. The vectors e_i $(i = 1, 2, 3)$ in $\mathbf{R}^2$ are given by

$$e_1 = (1, 0), \quad e_2 = (0, 1) \quad \text{and} \quad e_3 = (1, 1).$$

The difference operators ∇_i are given by the rule

$$\nabla_i f = f - f(\cdot - e_i), \quad \text{for } f\colon Z^2 \to \mathbf{R}.$$

The differential operators D_i are given by

$$D_i = \frac{\partial}{\partial x_i} \quad (i = 1, 2) \qquad \text{and} \qquad D_3 = D_1 + D_2.$$

Moreover, we denote by π_k the space of all polynomials in two variables of total degree no more than k. Let us recall the definition of box splines from [**BH**]. For a sequence $\Xi = (\xi_i)_1^n$ of vectors in $\mathbf{R}^n$, the box spline M_Ξ is defined to be the distribution given by

$$\langle M_\Xi, \phi \rangle = \int_{[0,1]^n} \phi \left(\sum_{i=1}^n \lambda(i) \xi_i \right) d\lambda$$

for any C^∞-locally supported function ϕ. If Ξ consists of e_1, e_2 and e_3, repeated r, s and t times, respectively, then we write $M_{r,s,t}$ instead of M_Ξ. In particular

$$M_{r,s,0}(x_1, x_2) = M_r(x_1)M_s(x_2).$$

Here M_r is the usual (univariant) B-spline of order r at the knot sequence $0, 1, \ldots, r$:

$$M_r(x) = [0, 1, \ldots, r](\cdot - x)_+^{r-1},$$

where $[\rho_0, \rho_1, \ldots, \rho_r]f$ means the divided difference of f at points $\rho_0, \rho_1, \ldots, \rho_r$.

Now we can formulate our collection $\Phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}$. Our candidates are $M_{2,0,1}, M_{2,1,0}, M_{0,2,1}$ and $M_{1,2,0}$. But we want to modify them to suit our purpose. First, for a technical reason which will be clear later, we want to shift those splines. Thus we define the shift operator τ by

$$\tau f := f(\cdot + e_3).$$

Second, and more importantly, those splines are not in the space $W^{1,2}$. So we have to smooth them. This can be done by convolving them with a sufficiently smooth function. We should be careful in doing so, however, because we want to keep Φ from satisfying $(\mathrm{ii})_1$. Thus we introduce the convolution operator σ as follows:

$$\begin{aligned}\sigma f &:= \int f(x_1 - t, x_2 - t) 2M_1\left(2t + \frac{1}{2}\right) dt \\ &= 2\int_{-1/4}^{1/4} f(x_1 - t, x_2 - t) dt.\end{aligned}$$

Note that σ and τ commute with each other. Let

$$\begin{aligned}\phi_1 &:= \sigma\tau(M_{2,0,1}), \quad \phi_2 := \sigma\tau(M_{2,1,0}), \\ \phi_3 &:= \sigma\tau(M_{0,2,1}), \quad \phi_4 := \sigma\tau(M_{1,2,0}).\end{aligned}$$

Obviously, ϕ_l $(l = 1, \ldots, 4)$ are in $W^{1,2}$. Our result can be stated in the following

THEOREM. *The collection Φ satisfies* $(\mathrm{i})_1$ *but not* $(\mathrm{ii})_1$.

PROOF. To prove the first assertion, we start with the box spline $M_{1,1,1}$. This spline is well known to finite element analysts as the standard linear element, which was first introduced by Courant [C]. The spline $M_{1,1,1}$ has the hexagon $\{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3; 0 \le \lambda_i \le 1\}$ as its support, and $M_{1,1,1}(1,1) = 1$. Let $\phi = \sigma\tau(M_{1,1,1})$. We claim

$$\sum_{j \in \mathbf{Z}^2} p(j)\phi(\cdot - j) = p \quad \text{for any } p \in \pi_1. \tag{3}$$

To this end, we recall the following elementary properties of box splines (see [**BH**]):

$$\sum M_\Xi(\cdot - j) = 1$$

and

$$D_i\left(\sum a(j) M_\Xi(\cdot - j)\right) = \sum (\nabla_i a)(j) M_{\Xi \setminus e_i}(\cdot - j) \quad \text{if } e_i \in \Xi. \tag{4}$$

For $p \in \pi_1$, $D_i p = \nabla_i p$ $(i = 1, 2)$ are constants; hence

$$D_i\left(\sum p(j) M_{1,1,1}(\cdot - j + e_3)\right) = \sum (\nabla_i p)(j) M_{\{e_1,e_2,e_3\}\setminus e_i}(\cdot - j + e_3) = D_i p.$$

Moreover,

$$\sum p(j)M_{1,1,1}(0-j+e_3)=p(0).$$

This shows that

$$\sum p(j)(\tau M_{1,1,1})(\cdot-j)=p \quad \text{for } p\in\pi_1.$$

Let the convolution operator σ act on both sides of this equality. We have

$$\sum p(j)(\sigma\tau M_{1,1,1})(\cdot-j)=\sigma p.$$

It is easily seen that $\sigma p=p$ for any $p\in\pi_1$. This proves our claim (3).

Now $\{\phi\}$ satisfies hypothesis $(\mathrm{ii})_1$ of Theorem 1, and $(\mathrm{ii})_1$ implies $(\mathrm{i})_1$. Thus for each $u\in W^{2,2}$, we can find $w\colon \mathbf{Z}^2\to\mathbf{R}$ such that

$$\left|u-\sum w(j)h^{-1}\phi(\cdot/h-j)\right|_{s,2}\le \operatorname{const} h^{2-s}|u|_{2,2}, \qquad s=0,1. \tag{5}$$

and $\|w\|_2\le \operatorname{const}\|u\|_2$. The sum in the left-hand side of (5) is denoted by v.

We want to express v as an *infinite* linear combination of $\phi_{i,j}^h$. This is possible because of (4). Suppose $\gamma\colon\mathbf{Z}^2\to\mathbf{R}$ satisfies $\nabla_1\gamma=w$. Then it follows from (4) that

$$\begin{aligned}\sum w(j)M_{1,1,1}(\cdot-j)&=\sum(\nabla_1\gamma)(j)M_{1,1,1}(\cdot-j)\\ &=D_1\left(\sum\gamma(j)M_{2,1,1}(\cdot-j)\right)=(D_3-D_2)\left(\sum\gamma(j)M_{2,1,1}(\cdot-j)\right)\\ &=\sum(\nabla_3\gamma)(j)M_{2,1,0}(\cdot-j)-\sum(\nabla_2\gamma)(j)M_{2,0,1}(\cdot-j).\end{aligned}$$

Recall that

$$\phi=\sigma\tau(M_{1,1,1}),\quad \phi_1=\sigma\tau(M_{2,0,1})\quad\text{and}\quad \phi_2=\sigma\tau(M_{2,1,0}).$$

We obtain

$$\sum w(j)\phi_j^h=\sum(\nabla_3\gamma)(j)\phi_{2,j}^h-\sum(\nabla_2\gamma)(j)\phi_{1,j}^h.$$

Thus v can be expressed as an infinite linear combination of $\phi_{1,j}^h$ and $\phi_{2,j}^h$. But we have trouble in keeping the coefficient sequence $\nabla_2\gamma$ bounded in l^2. To settle this trouble we shall use Fourier series. From now on, the letter i will be reserved for the imaginary unit $\sqrt{-1}$. For any $a\in l^2(\mathbf{Z}^2)$, we denote by $\hat a$ the Fourier series given by

$$\hat a(\xi):=\sum a(j)e^{2\pi i j\cdot\xi},\qquad \xi\in\mathbf{R}^2,$$

where $j\cdot\xi=j_1\xi_1+j_2\xi_2$ is the inner product of j and ξ. We have the following elementary property:

$$(\nabla_k a)\hat{}=\sum(a(j)-a(j-e_k))e^{2\pi i j\cdot\xi}=(1-\exp(2\pi i\xi_k))\hat a,\qquad k=1,2. \tag{6}$$

Let

$$g_k(\xi):=1-\exp(2\pi i\xi_k),\qquad k=1,2.$$

Then (6) tells us that

$$(\nabla_2 a)\hat{}=g_2\hat a=(g_2/g_1)(\nabla_1 a)\hat{}. \tag{7}$$

Let Q be the square $[0,1]\times[0,1]$. By Parseval's identity,

$$\|\hat w\|_{2,Q}:=\left(\int_Q|\hat w|^2\right)^{1/2}=\|w\|_2.$$

Thus if $\nabla_1\gamma = w$, then

$$\|\nabla_2\gamma\|_2 = \|(\nabla_2\gamma)\hat{}\|_{2,Q} = \|(g_2/g_1)(\nabla_1\gamma)\hat{}\|_{2,Q} = \|(g_2/g_1)\hat{w}\|_{2,Q}.$$

Since g_2/g_1 is unbounded, the norm $\|(g_2/g_1)\hat{w}\|_{2,Q}$ may be unbounded in general. Our scheme is to decompose w into a sum of two sequences a and b. For a, we seek α such that $\nabla_1\alpha = a$ and $\|\nabla_2\alpha\|_2 \le \|w\|_2$, and for b, we find β such that $\nabla_2\beta = b$ and $\|\nabla_1\beta\|_2 \le \|w\|_2$. This decomposition of w can be done in the following way: Introduce the set

$$E := \{(\xi_1, \xi_2) \in Q;\ |g_1(\xi)| \ge |g_2(\xi)|\},$$

and its characteristic function

$$\chi_E(\xi) := \begin{cases} 1 & \text{if } \xi \in E, \\ 0 & \text{if } \xi \in Q\backslash E. \end{cases}$$

Decompose $\hat{w}$ into $\hat{w}\chi_E + \hat{w}(1-\chi_E)$. Obviously,

$$\|\hat{w}\chi_E\|_{2,Q} \le \|\hat{w}\|_{2,Q} \quad \text{and} \quad \|\hat{w}(1-\chi_E)\|_{2,Q} \le \|\hat{w}\|_{2,Q}.$$

Hence we can expand $\hat{w}\chi_E$ and $\hat{w}(1-\chi_E)$ in Fourier series:

$$\hat{w}\chi_E(\xi) = \sum a(j)e^{2\pi i j\cdot\xi}, \quad \hat{w}(1-\chi_E)(\xi) = \sum b(j)e^{2\pi i j\cdot\xi}$$

with $\|a\|_2, \|b\|_2 \le \|\hat{w}\|_{2,Q} = \|w\|_2$. It follows that

$$\sum w(j)e^{2\pi i j\cdot\xi} = \hat{w}(\xi) = \sum (a(j)+b(j))e^{2\pi i j\cdot\xi}.$$

By the uniqueness of Fourier series expansion, we have $w = a + b$. Since $\hat{a} = \hat{w}\chi_E$, and since $|g_2(\xi)| \le |g_1(\xi)|$ for $\xi \in E$, we have

$$|(g_2/g_1)\hat{a}(\xi)| \le |\hat{a}(\xi)| \quad \text{for } \xi \in Q,$$

and therefore we can find $c \in l^2(\mathbf{Z}^2)$ such that

$$\hat{c} = (g_2/g_1)\hat{a}. \tag{8}$$

In addition,

$$\|c\|_2 = \|\hat{c}\|_{2,Q} \le \|\hat{a}\|_{2,Q} = \|a\|_2 \le \|w\|_2.$$

Furthermore, (8) is equivalent to

$$g_1\hat{c} = g_2\hat{a}, \quad \text{or} \quad (\nabla_1 c)\hat{} = (\nabla_2 a)\hat{};$$

hence $\nabla_1 c = \nabla_2 a$.

Having obtained a and c, we can find (uniquely) a sequence $\alpha\colon \mathbf{Z}^2 \to \mathbf{R}$ such that $\nabla_1\alpha = a$ and

$$\alpha(0, j_2) = \begin{cases} \sum_{l=1}^{j_2} c(0,l) & \text{for } j_2 > 0, \\ 0 & \text{for } j_2 = 0, \\ -\sum_{l=j_2+1}^{0} c(0,l) & \text{for } j_2 < 0. \end{cases} \tag{9}$$

Then

$$\nabla_1(\nabla_2\alpha) = \nabla_2(\nabla_1\alpha) = \nabla_2 a = \nabla_1 c.$$

But (9) tells us that

$$\nabla_2\alpha(0, j_2) = c(0, j_2) \quad \text{for all } j_2 \in \mathbf{Z};$$

therefore $\nabla_2 \alpha = c$. Furthermore

$$\nabla_3 \alpha = \alpha - \alpha(\cdot - e_3) = \alpha - \alpha(\cdot - e_2) + \alpha(\cdot - e_2) - \alpha(\cdot - e_3)$$
$$= \nabla_2 \alpha + \nabla_1 \alpha(\cdot - e_2) = \nabla_2 \alpha + a(\cdot - e_2);$$

hence

$$\|\nabla_3 \alpha\|_2 \le 2\|a\|_2 \le 2\|w\|_2.$$

To sum up, we can find $\alpha\colon \mathbf{Z}^2 \to \mathbf{R}$ such that

$$\|\nabla_2 \alpha\|_2 \le \|w\|_2, \quad \|\nabla_3 \alpha\|_2 \le 2\|w\|_2$$

and

$$\sum a(j)\phi_j^h = \sum (\nabla_3 \alpha)(j)\phi_{2,j}^h - \sum (\nabla_2 \alpha)(j)\phi_{1,j}^h.$$

Similarly, we can find $\beta\colon \mathbf{Z}^2 \to \mathbf{R}$ such that

$$\|\nabla_1 \beta\|_2 \le \|w\|_2, \quad \|\nabla_3 \beta\|_2 \le 2\|w\|_2$$

and

$$\sum b(j)\phi_j^h = \sum (\nabla_3 \beta)(j)\phi_{4,j}^h - \sum (\nabla_1 \beta)(j)\phi_{3,j}^h.$$

Recall that

$$v = \sum w(j)\phi_j^h = \sum a(j)\phi_j^h + \sum b(j)\phi_j^h.$$

From (5) we observe that

$$|u - v|_{s,2} \le \text{const}\, h^{2-s}|u|_{2,2}, \qquad s = 0, 1.$$

Thus we have proved that the collection $\Phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ satisfies $(\mathrm{i})_1$.

It remains to show that Φ does not satisfy $(\mathrm{ii})_1$. This will be done by contradiction. Suppose to the contrary that there is a *finite* linear combination Ω of $\phi_1, \ldots, \phi_4$ and their translates $\phi_{l,j}$ such that

> for $|\alpha| \le 1$, $\sum_{j \in \mathbf{Z}^2} j^\alpha \Omega(x-j)$ is a polynomial in x_1, x_2 with leading term x^α.

Assume

$$\Omega = \sum_{l=1}^{4} \sum_{m \in I} c_{l,m} \phi_l(\cdot - m), \tag{10}$$

where I is a *finite* subset of $\mathbf{Z}^2$. We want to draw a contradition from the above assumption. Let us first take a closer look at the function ϕ_1. By straightforward computation we have

$$\phi_1(x_1, x_2) = M_2(x_1 - x_2 + 1)\psi(x_2)$$

with

$$\psi(x_2) = \begin{cases} 2(x_2 + 5/4) & \text{for } -5/4 \le x_2 \le -3/4, \\ 1 & \text{for } -3/4 \le x_2 \le -1/4, \\ -2(x_2 - 1/4) & \text{for } -1/4 \le x_2 \le 1/4, \\ 0 & \text{for } x_2 < -4/5 \text{ or } x_2 > 1/4. \end{cases}$$

This motivates us to introduce the following functional F:

$$Fg := \lim_{\varepsilon \to +0} \left[D_3 g\left(\frac{1}{4} + \varepsilon, \frac{1}{4} + \varepsilon\right) - D_3 g\left(\frac{1}{4} - \varepsilon, \frac{1}{4} - \varepsilon\right)\right] \quad \text{for } g \in W^{1,\infty}(\mathbf{R}^2).$$

Just by computation, we obtain

$$F(\phi_1(\cdot - j)) = \begin{cases} 2 & \text{if } j = (0,0), \\ -2 & \text{if } j = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$F\left(\sum j_2 \phi_1(\cdot - j)\right) = -2.$$

By (4) we have

$$\begin{aligned} \sum j_2 M_{2,1,0}(\cdot - j) &= D_3 \sum (j_2(j_2+1)/2) M_{2,1,1}(\cdot - j) \\ &= (D_1 + D_2) \sum (j_2(j_2+1)/2) M_{2,1,1}(\cdot - j) \\ &= \sum j_2 M_{2,0,1}(\cdot - j); \end{aligned}$$

hence

$$F\left(\sum j_2 \phi_2(\cdot - j)\right) = F\left(\sum j_2 \phi_1(\cdot - j)\right) = -2. \tag{11}$$

By an argument similar to that used in proving (3), we observe that

$$\sum_j j_2 \phi_l(\cdot - j) \text{ is in } \pi_1 \text{ for } l = 3 \text{ or } 4.$$

Therefore

$$F\left(\sum_j j_2 \phi_l(\cdot - j)\right) = 0, \qquad l = 3 \text{ or } 4. \tag{12}$$

Now we consider the function $q := \sum j_2 \Omega(\cdot - j)$. On the one hand, $q \in \pi_1$; hence $F(q) = 0$. On the other hand, by (10),

$$q = \sum_j j_2 \Omega(\cdot - j) = \sum_j \sum_m \sum_{l=1}^{4} c_{l,m} j_2 \phi_l(\cdot - m - j),$$

hence by (11) and (12),

$$F(q) = \sum_m \sum_{l=1}^{4} c_{l,m} F\left(\sum j_2 \phi_l(\cdot - m - j)\right) = -2 \sum_{l=1}^{2} \sum_m c_{l,m}.$$

This shows that

$$\sum_{l=1}^{2} \sum_m c_{l,m} = 0.$$

Similarly, one can show

$$\sum_{l=3}^{4} \sum_m c_{l,m} = 0.$$

Finally, since $\sum \phi_l(\cdot - j) = 1$, we get

$$\sum \Omega(\cdot - j) = \sum_{l=1}^{4} \sum_m c_{l,m} \left(\sum_j \phi_l(\cdot - m - j)\right) = \sum_{l=1}^{4} \sum_m c_{l,m} = 0.$$

This shows that Φ does not satisfy $(\mathrm{ii})_1$. Our proof is complete.

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