

CONTINUITY OF THE FENCHEL TRANSFORM OF CONVEX FUNCTIONS

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ABSTRACT. Given a separated dual system (E, E') , the Fenchel transform determines a pairing of the convex functions on E with the convex functions on E' . This operation is shown to have a continuity property. The result implies that the minimum set of a convex function varies in an upper-semicontinuous way with the function's conjugate. Several convergence concepts for convex functions are discussed. It is shown for each of the two most useful that the Fenchel transform is not a homeomorphism.

Let (E, E') be a separated dual system. Denote by $\Gamma(E)$ the class of convex, $\sigma(E, E')$ lower-semicontinuous $\overline{\mathbf{R}}$ -valued functions which do not take both the values $-\infty, +\infty$. These are the functions which are upper envelopes of collections of affine functions $x \rightarrow \langle x, x' \rangle - a$, where $x' \in E'$ and $a \in \mathbf{R}$. The class $\Gamma(E')$ is defined symmetrically.

The Fenchel transform (conjugate) of $f \in \Gamma(E)$ is the function $f^* \in \Gamma(E')$ defined by $f^*(x') = \sup\{\langle x, x' \rangle - f(x) \mid x \in E\}$. This determines a bijection of $\Gamma(E)$ onto $\Gamma(E')$, and, defining the conjugate of $f^* \in \Gamma(E')$ symmetrically, one has $f^{**} = f$.

This paper studies the continuity of the Fenchel transform. The interest is in topologies on $\Gamma(E')$ for which the multifunction $f^* \rightarrow \operatorname{argmin} f^*$ has upper-semicontinuity properties, the ultimate objective being to provide conditions on sequences or nets of functions $f_i \in \Gamma(E)$ which, when satisfied, will ensure the convergence of the minimum sets of their conjugates.

The continuity of the Fenchel transform was first established by Wijsman [9] in finite-dimensional spaces. At the same time Walkup and Wets [7] showed that in reflexive Banach spaces the polarity operation for closed convex cones is an isometry, a fact which implies the continuity of the Fenchel transform in finite-dimensional spaces (see Wets [8]). The continuity in reflexive Banach spaces was established by Mosco [3]. Joly [1] then obtained the same result and several generalizations by another argument.

Our point of departure is the work of Joly. In §1 we review his arguments, leading to a continuity theorem in Fréchet spaces.

In §2 the concept of convergence in $\Gamma(E)$ will be strengthened and a continuity theorem will be proven without restrictions on the dual system (E, E') . The result

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seems suitable for applications, but it is not symmetric. It will be demonstrated by examples that the convergence concept for $\Gamma(E)$ cannot be weakened nor that for $\Gamma(E')$ strengthened.

The results of convex analysis which will be used can be found in Rockafellar [6].

1. Topologies on $\Gamma(E)$ and $\Gamma(E')$. We briefly review the notation and results of Joly [1]. Let σ be the weak topology $\sigma(E, E')$, τ the Mackey topology $\tau(E, E')$, and $\sigma' = \sigma(E', E)$, $\tau' = \tau(E', E)$. Joly defines a topology \mathcal{T}_τ on $\Gamma(E)$ in which a net (f_i) converges to a point f iff

$$(1.1) \quad \inf_{x \in U} f(x) \geq \limsup_i \inf_{x \in U} f_i(x)$$

for each $U \in \tau$. A topology $\mathcal{T}_{\tau'}$ on $\Gamma(E')$ is defined symmetrically.

Denote by \mathcal{T}_τ^* the coarsest topology on $\Gamma(E)$ which renders continuous the Fenchel transform from $\Gamma(E)$ to $(\Gamma(E'), \mathcal{T}_{\tau'})$. Convergence of a net (f_i) in \mathcal{T}_τ^* is characterized by having

$$(1.2) \quad \inf_{x' \in U'} f^*(x') \geq \limsup_i \inf_{x' \in U'} f_i^*(x')$$

for each $U' \in \tau'$. Define $\mathcal{T}_{\tau'}^*$ on $\Gamma(E')$ symmetrically, and denote by $\mathcal{T}_{\tau\tau'}$ (resp. $\mathcal{T}_{\tau'\tau}$) the least upper bound of the topologies \mathcal{T}_τ and $\mathcal{T}_{\tau'}^*$ (resp. $\mathcal{T}_{\tau'}$ and \mathcal{T}_τ^*).

Convergence of a net (f_i) in $\mathcal{T}_{\tau\tau'}$ and convergence of the conjugates (f_i^*) in $\mathcal{T}_{\tau'\tau}$ are both defined by having (1.1) and (1.2) hold simultaneously. Certainly the Fenchel transform is a homeomorphism of $(\Gamma(E), \mathcal{T}_{\tau\tau'})$ onto $(\Gamma(E'), \mathcal{T}_{\tau'\tau})$ [1, Corollarie, p. 423].

This establishes the notation. The work to be done is to characterize the convergence of a net (f_i) in $\mathcal{T}_{\tau\tau'}$ in terms of directly verifiable conditions on the functions f_i and to interpret the convergence of the functions f_i^* in $\mathcal{T}_{\tau'\tau}$.

Joly's insight is to introduce a topology \mathcal{A}_τ^* on $\Gamma(E)$ in which a net (f_i) converges to a point f iff

$$(1.3) \quad \inf_{x' \in U'} (f + \psi_K)^*(x') \geq \limsup_i \inf_{x' \in U'} (f_i + \psi_K)^*(x')$$

for each $U' \in \tau'$ and each weakly compact disk $K \subset E$ (the symbol ψ_K denotes the function which takes the value 0 on K and $+\infty$ elsewhere). A topology $\mathcal{A}_{\tau'}^*$ on $\Gamma(E')$ is defined symmetrically.

Condition (1.3) is readily seen to be equivalent to having

$$(1.4) \quad f(x) \leq \liminf_i f_i(x_i)$$

for each $x \in E$, each subnet (f_i) of the net (f_i) and each net (x_i) which is weakly convergent to x and contained in a weakly compact disk [1, Remarque, p. 430].

Moreover, Joly observes that $\mathcal{A}_\tau^* \subset \mathcal{T}_\tau^*$ and $\mathcal{A}_{\tau'}^* \subset \mathcal{T}_{\tau'}^*$ [1, Proposition 7]. From the latter relation and the analogue of (1.4) we see that the set

$$\{(f^*, x') \mid x' \in \operatorname{argmin} f^*\} \cap [\Gamma(E') \times K]$$

is $(\mathcal{T}_{\tau\tau'} \times \sigma')$ -closed for each weakly compact disk $K \subset E'$. This upper-semicontinuity result seems suitable for applications since in practice one is likely to use the Banach-Alaoglu Theorem to verify the existence of a weak limit point of minimizing arguments for the functions f_i^* . Certainly it is satisfactory if (E', τ') is complete.

It remains to characterize the convergence of the f_l in $\mathcal{T}_{\tau\tau'}$, specifically to state conditions on the f_l which imply (1.2).

Denote by $\mathcal{A}_{\tau\tau'}$ (resp. $\mathcal{A}_{\tau'\tau}$) the least upper bound of the topologies \mathcal{T}_{τ} and \mathcal{A}_{τ}^* (resp. $\mathcal{T}_{\tau'}$ and $\mathcal{A}_{\tau'}^*$).

Joly observes that the relative \mathcal{A}_{τ}^* and \mathcal{T}_{τ}^* topologies coincide on subsets of $\Gamma(E)$ for which (the constant function) $+\infty$ is not an \mathcal{A}_{τ}^* -adherent point [1, Proposition 8]. Thus a net converges in $\mathcal{T}_{\tau\tau'}$ if it satisfies (1.1) and (1.4) and if its \mathcal{A}_{τ}^* -adherence does not include $+\infty$. In certain cases one may deduce from convergence in \mathcal{T}_{τ} , i.e. (1.1), that the \mathcal{A}_{τ}^* -adherence does not include $+\infty$ (the convergence in \mathcal{A}_{τ}^* has no implications in this regard since each \mathcal{A}_{τ}^* open set includes $+\infty$).

PROPOSITION [1]. *Suppose E is a Fréchet space with dual E' . Then a sequence (f_n) from $\Gamma(E)$ converges to $f \neq +\infty$ in $\mathcal{A}_{\tau\tau'}$ if and only if the sequence (f_n^*) converges to f^* in $\mathcal{T}_{\tau'\tau}$.*

PROOF. The claim is that convergence to $f \neq +\infty$ in $\mathcal{A}_{\tau\tau'}$ implies convergence in \mathcal{T}_{τ}^* . It suffices to show that convergence in \mathcal{T}_{τ} implies that the constant function $+\infty$ is not in the \mathcal{A}_{τ}^* -adherence of the sequence (f_n) . Choosing x such that $f(x) < \infty$, condition (1.1) implies the existence of a sequence (x_n) which is τ -convergent to x and satisfies $\limsup_n f_n(x_n) < \infty$. Since E is complete, the weakly closed disked hull of the sequence is weakly compact, and the result follows. \square

In other cases a different method will be necessary.

EXAMPLE 1. Denote by E the space of sequences $x = (\xi_i)$ with at most a finite number of nonzero terms, with the norm $\|x\| = \sum |\xi_i|$. Let x_n be the sequence with all terms zero except the n th, which is $1/n$. Let $f_n = \psi_{\{x_n\}}$ and $f = \psi_{\{0\}}$. The sequence (f_n) converges to f in \mathcal{T}_{τ} and \mathcal{A}_{τ}^* ; in fact, (1.4) holds without the restriction that the net be contained in a weakly compact disk. Yet the constant function $+\infty$ is in the \mathcal{A}_{τ}^* -adherence of the sequence (f_n) . This follows from the fact that each weakly compact disk in E is finite dimensional (Kelley-Namioka [2, Problem 5.17.I]); so for such a disk K we eventually have $f_n + \psi_K \equiv +\infty$.

2. A continuity Theorem and examples. We will say that a net (f_l) is M_{σ} -convergent to f in $\Gamma(E)$ if (1.4) holds for each $x \in E$, each subnet (f_{l_i}) of the net (f_l) , and each net (x_i) which is weakly convergent to x . This and the concept of \mathcal{A}_{τ}^* -convergence as well are versions of a condition stated by Mosco [3] (for sequences and subsequences in a reflexive Banach space).

The net will be said to be $M_{\tau\sigma}$ -convergent if it is M_{σ} -convergent and also convergent in the topology \mathcal{T}_{τ} . It should be noted that this does not agree with the definition of Joly [1].

Example 1 shows that a sequence may be $M_{\tau\sigma}$ -convergent but yet include $+\infty$ in its \mathcal{A}_{τ}^* -adherence. We show by a direct argument that $M_{\tau\sigma}$ -convergent nets are convergent in the topology $\mathcal{T}_{\tau\tau}$.

THEOREM. *Suppose a net $(f_l, l \in L)$ is $M_{\tau\sigma}$ -convergent in $\Gamma(E)$ to $f \neq +\infty$. Then the net $(f_l^*, l \in L)$ converges to f^* in the topology $\mathcal{T}_{\tau'\tau}$.*

PROOF. We must show that (1.2) is satisfied. If $f \equiv -\infty$ then $f^* \equiv +\infty$ and (1.2) holds trivially. If $f \not\equiv -\infty$ then the M_σ -convergence implies that eventually the f_l are different from the constant function $-\infty$. Furthermore, since $f \not\equiv +\infty$, the convergence in the topology \mathcal{T}_τ implies that eventually $f_l \not\equiv +\infty$. Hence we can and will assume that none of the f_l nor f is one of the constant functions $-\infty, +\infty$. This implies that the same is true of the f_l^* and f^* , so we are dealing exclusively with what is termed in [6] “proper” convex functions.

Fix $U' \in \tau'$. There is no loss in assuming the existence of x'_0 in U' satisfying $f^*(x'_0) < \infty$. Let K be a weakly compact disk in E whose polar $K^0 \equiv \{x' \in E' \mid \psi_K^*(x') \leq 1\}$ satisfies $\{x'_0\} - K^0 \subset U'$.

Step 1. We will show that there is eventually $x'_l \in E'$ satisfying $\psi_K^*(x'_0 - x'_l) < 1$ and $f_l^*(x'_l) < \infty$. Denote by D (resp. D_l) the weak closure of the (convex) set $\{x' \in E' \mid f^*(x') < \infty\}$ (resp. $\{x' \in E' \mid f_l^*(x') < \infty\}$).

First we claim that

$$(2.1) \quad \inf_{x \in E} [\psi_{D_l}^*(x) + \psi_K(x) - \langle x, x'_0 \rangle] = \sup_{x' \in E'} [-\psi_{D_l}(x') - \psi_K^*(x'_0 - x')].$$

To see this define $\phi: E \rightarrow \bar{\mathbf{R}}$ by

$$\phi(u) = \inf_{x \in E} [\psi_{D_l}^*(x) + \psi_K(x - u) - \langle x - u, x'_0 \rangle].$$

The function ϕ is convex, lower-semicontinuous and never takes the value $-\infty$. Thus $\phi = \phi^{**}$. The left-hand side of (2.1) is $\phi(0)$. Also for any $x' \in E'$ we have

$$\begin{aligned} \sup_{u \in E} [\langle u, x' \rangle - \phi(u)] &= \sup_{\substack{u \in E \\ x \in E}} [\langle u, x' \rangle - \psi_{D_l}^*(x) - \psi_K(x - u) + \langle x - u, x'_0 \rangle] \\ &= \sup_{x \in E} [\langle x, x' \rangle - \psi_{D_l}^*(x) + \sup_{u \in E} \{\langle x - u, x'_0 - x' \rangle - \psi_K(x - u)\}] \\ &\equiv \sup_{x \in E} [\langle x, x' \rangle - \psi_{D_l}^*(x) + \psi_K^*(x'_0 - x')] = \psi_{D_l}(x') + \psi_K^*(x'_0 - x'). \end{aligned}$$

The right-hand side of (2.1) is therefore $\phi^{**}(0)$, so the equality holds.

Both sides of (2.1) are finite since the left-hand side is larger than $-\infty$ and the right-hand side is nonpositive. Hence it may be rewritten as

$$\inf_{x \in K} [\psi_{D_l}^*(x) - \langle x, x'_0 \rangle] + \inf_{x' \in D_l} [\psi_K^*(x'_0 - x')] = 0.$$

Choose for each l an $x_l \in K$ and an $x'_l \in D_l$ satisfying

$$\psi_{D_l}^*(x_l) - \langle x_l, x'_0 \rangle + \psi_K^*(x'_0 - x'_l) < \frac{1}{2}.$$

Denote by M the set of $l \in L$ such that $\psi_{D_l}^*(x_l) - \langle x_l, x'_0 \rangle < -\frac{1}{2}$. It suffices to show that M is not cofinal in L . Suppose for the sake of the argument that it is.

Since the net $(x_l, l \in M)$ is contained in K , there is a subnet $(x_i, i \in I)$ weakly convergent to an $x \in E$. We will show that $\langle x, x'_0 \rangle \leq \liminf_i \psi_{D_i}^*(x_i)$, contradicting the definition of M .

Notice first that $\langle x, x'_0 \rangle \leq \psi_D^*(x)$, since $x'_0 \in D$. Also, according to Rockafellar [5, Corollary 3c], $\psi_D^*(x) = \sup\{f(x + y) - f(y) \mid f(y) < \infty\}$.

Choose an arbitrary $y \in E$ satisfying $f(y) < \infty$.

Denote by \mathcal{U} the family of τ -open sets containing y , and direct it by inclusion. Let A denote the set of $(U, r, i) \in \mathcal{U} \times (0, \infty) \times I$ such that

$$\inf_{z \in U} f_i(z) \leq f(y) + r^{-1}.$$

By virtue of the convergence in \mathcal{T}_τ this inequality holds eventually in i for each (U, r) . Thus A is directed by the product order, and the projection of A is cofinal in I .

For each $a = (U, r, i) \in A$ set $x_a = x_i, f_a = f_i, D_a = D_i$, and choose $y_a \in U$ in the following way: if $\inf\{f_a(z) | z \in U\} = -\infty$, let $f_a(y_a) \leq -r$; otherwise let $f_a(y_a) \leq \inf\{f_a(z) | z \in U\} + r^{-1}$. By this construction we have

$$(2.2) \quad \limsup_a f_a(y_a) \leq \inf_a \max\{-r, f(y) + r^{-1}\} \leq f(y).$$

The net $(x_a, a \in A)$ is weakly convergent to x , and the net $(y_a, a \in A)$ is weakly convergent to y . We therefore obtain from the M_σ -convergence and (2.2) that

$$f(x + y) - f(y) \leq \liminf_a [f_a(x_a + y_a) - f_a(y_a)].$$

By virtue of (2.2) we eventually have $f_a(y_a) < \infty$. Thus again applying [5, Corollary 3c] we obtain

$$f(x + y) - f(y) \leq \liminf_a \psi_{D_a}^*(x_a) \equiv \liminf_i \psi_{D_i}^*(x_i),$$

and since y was chosen arbitrarily, this completes the argument.

Step 2. The function $x' \rightarrow \psi_{K^0}(x'_0 - x')$ is τ' -continuous on $\{x' | \psi_{K^0}^*(x'_0 - x') < 1\}$ so it follows from the preceding step and a version of Fenchel's Duality Theorem (see Rockafellar [4]) that eventually

$$\inf_{x' \in E'} [f_l^*(x') + \psi_{K^0}(x'_0 - x')] = \max_{x \in E} [\langle x, x'_0 \rangle - \psi_{K^0}^*(x) - f_l(x)].$$

For each such l choose $x_l \in E$ satisfying

$$\inf_{x' \in \{x'_0\} - K^0} f_l^*(x') = \langle x_l, x'_0 \rangle - \psi_{K^0}^*(x_l) - f_l(x_l).$$

Given an arbitrary $\varepsilon > 0$, denote by M the set of such l satisfying

$$(2.3) \quad f^*(x'_0) + \varepsilon \leq \langle x_l, x'_0 \rangle - \psi_{K^0}^*(x_l) - f_l(x_l).$$

Since the right-hand side of (2.3) majorizes $\inf\{f_l^*(x') | x' \in U'\}$, the proof will be complete if we show that M is not cofinal in L . Suppose for the sake of the argument that it is.

We will deduce that the net $(x_l, l \in M)$ has a subnet (x_l) which is weakly convergent to an $x \in E$. By virtue of the M_σ -convergence we must have

$$\langle x, x'_0 \rangle - f(x) \geq \limsup_i [\langle x_l, x'_0 \rangle - f_l(x_l)].$$

But $f^*(x'_0) \geq \langle x, x'_0 \rangle - f(x)$ by definition, and $\psi_{K^0}^* \geq 0$, so this will contradict the definition of M and complete the proof.

It is convenient to rewrite (2.3) as

$$(2.4) \quad \psi_{K^0}^*(x_l) \leq \langle x_l, x'_0 \rangle - f_l(x_l) + s,$$

where $s = -f^*(x'_0) - \varepsilon \neq \pm\infty$ (s is less than $+\infty$ since $f \not\equiv +\infty$). Choose an $r > 0$ and let $J_r = \{l \in M \mid \langle x_l, x'_0 \rangle - f_l(x_l) > r\}$. If J_r is not cofinal in M then, by virtue of (2.4), the net $(x_l, l \in M)$ is eventually within a weakly compact set, and we are done. Suppose conversely that J_r is cofinal in M .

Fix $y \in E$ such that $f(y) < \infty$. As in Step 1 use the \mathcal{F}_τ -convergence to construct a directed set A , an isotone map $v: A \rightarrow J_r$, and a net $(y_a, a \in A)$ which is τ -convergent to y and satisfies $\limsup_a f_a(y_a) \leq f(y)$, where we set $f_a = f_{v(a)}$.

Define $x_a = x_{v(a)}$, $\lambda_a = r \cdot [\langle x_a, x'_0 \rangle - f_a(x_a)]^{-1}$ and $z_a = \lambda_a x_a + (1 - \lambda_a) y_a$. By the convexity of the f_a we have

$$\langle z_a, x'_0 \rangle - f_a(z_a) \geq r + (1 - \lambda_a) [\langle y_a, x'_0 \rangle - f_a(y_a)].$$

Hence

$$(2.5) \quad \liminf_a [\langle z_a, x'_0 \rangle - f_a(z_a)] \geq r + \min\{0, \langle y, x'_0 \rangle - f(y)\}.$$

Observe now that $\psi_{K^0}^*(\lambda_a x_a) = \lambda_a \psi_{K^0}^*(x_a) \leq r + \lambda_a s$ by virtue of (2.4). Hence the net $(\lambda_a x_a, a \in A)$ is weakly precompact. There is therefore a subnet of the net $(z_a, a \in A)$ which is weakly convergent to some $z \in E$.

From the M_σ -convergence and (2.5) we obtain

$$\langle z, x'_0 \rangle - f(z) \geq r + \min\{0, \langle y, x'_0 \rangle - f(y)\}.$$

But $f^*(x'_0) \equiv \sup\{\langle z, x'_0 \rangle - f(z) \mid z \in E\} < \infty$ by assumption and the choice of y did not depend on r , so there must exist an r such that J_r is not cofinal in M . \square

It would be useful if $M_{\tau\sigma}$ -convergence of functions in $\Gamma(E)$ implied $M_{\tau\sigma'}$ -convergence of the conjugates in $\Gamma(E')$, or if the Fenchel transform were a homeomorphism of $(\Gamma(E), \mathcal{A}_{\tau\tau})$ onto $(\Gamma(E'), \mathcal{A}_{\tau\tau})$. Unfortunately, neither of these is true in general.

EXAMPLE 1'. Define E , f and the sequence $(f_n, n \in N)$ as in Example 1. We will show that the sequence $(f_n^*, n \in N)$ is not M_σ -convergent to f^* . Denote by \mathcal{U}' the family of disked σ' neighborhoods of the origin in E' , and direct it by inclusion. Denote by A the set of $(U', n) \in \mathcal{U}' \times N$ for which there exists $x' \in U'$ satisfying $\langle x_n, x' \rangle \leq -1$. For each U' we know that there is eventually such an x' , since the polar of U' is finite dimensional. Hence A is directed by the product order. Select for each $a = (U', n) \in A$ some $x'_a \in U'$ with the above property, and set $f_a^* = f_n^*$. The net $(f_a^*, a \in A)$ is a subnet of the sequence $(f_n^*, n \in N)$, the net $(x'_a, a \in A)$ is weakly convergent to 0, yet $\liminf_a f_a^*(x'_a) \leq -1 < 0 \equiv f^*$.

EXAMPLE 2. Let E be the dual of the space of sequences defined before, with a topology compatible with the pairing, so that E' is now the space of sequences. Define a sequence (x_n) in E by setting $\langle x_n, x' \rangle = n\xi_n$ for each $x' = (\xi_1, \xi_2, \dots) \in E'$. Set $f = \psi_{\{0\}}$ and define $f_n(x_n) = -2n$, $f_n(x) = +\infty$ for $x \neq x_n$. The Mackey and weak topologies on E coincide since the weakly compact disks in E' are norm bounded finite-dimensional sets (see Kelley-Namioka [2, Problem 5.18.F]). Hence

the sequence (x_n) is τ -convergent to 0, and the sequence (f_n) is therefore \mathcal{T}_τ -convergent to f . It is also \mathcal{A}_τ^* -convergent since the sequence (x_n) eventually leaves the polar K of any norm neighborhood of the origin in E' , implying that $f_n + \psi_K \equiv +\infty$. However, the sequence (f_n^*) does not converge to f^* in the topology \mathcal{T}_τ : if U' is the unit ball in E' then

$$\inf_{x' \in U'} f_n^*(x') = \inf_{x' \in U'} [\langle x_n, x' \rangle + 2n] = n \rightarrow 0 \equiv f^*.$$

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