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COMPACT DERIVATIONS OF NEST ALGEBRAS

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ABSTRACT. In this paper we determine all the weakly compact derivations of a nest algebra. We also obtain necessary and sufficient conditions in order that a nest algebra admit compact derivations. Finally we prove that every compact derivation of a nest algebra $\mathscr{A}$ is the norm limit of finite-rank derivations.

1. Let $\mathscr{A}$ be a Banach algebra and let X be a Banach $\mathscr{A}$-module. By an X-valued derivation of $\mathscr{A}$ we mean a linear mapping δ: $\mathscr{A} \to \mathscr{X}$ with the property $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a \in \mathscr{A}$, $b \in \mathscr{B}$. The derivation δ is called compact if δ is a compact operator between the Banach space $\mathscr{A}$ and $\mathscr{X}$, and weakly compact if δ is a weakly compact operator from $\mathscr{A}$ to $\mathscr{X}$ (i.e. $\delta(\mathscr{A}_1)$ is relatively weakly compact in X, where $\mathscr{A}_1$ is the unit ball of $\mathscr{A}$ [4]). Let H be a complex Hilbert space, $B(H)$ the algebra of all bounded operators on H and $\mathscr{K}(H) = \mathscr{K}$ the set of compact operators on H. In [7] Johnson and Parrott investigated derivations of a von Neumann subalgebra of $B(H)$ with range contained in $\mathscr{K}$. They proved that in most cases such derivations are implemented by a compact operator. The general result was recently obtained by Popa [8] who proved that this is the case for all von Neumann subalgebras of $B(H)$. Such derivations are known to be weakly compact [1].

On the other hand, in a series of papers [1, 9, 10], C. A. Akemann, S. K. Tsui, and S. Wright have determined the structure of all compact and weakly compact $\mathscr{A}$-valued derivations of a C^*-algebra $\mathscr{A}$, and of all compact $B(H)$ valued derivations of a C^*-subalgebra of $B(H)$.

In this note we determine the structure of all $\mathscr{A}$-valued compact and weakly compact derivations of a nest algebra $\mathscr{A}$. In particular we prove that every compact derivation of a nest algebra $\mathscr{A}$ is the norm limit of finite rank derivations. We need the following result.

LEMMA 1 [6]. *Let H be an infinite dimensional Hilbert space. If δ is a compact derivation of $B(H)$ then $\delta \equiv 0$.*

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2. In this section we state and prove our results on nest algebras.

Let $\mathscr{N} \cap \mathscr{B}(H)$ be a complete nest of projections, i.e. a totally ordered set of orthogonal projections which is strongly closed. We denote by $\mathscr{A}$ the nest algebra

$$\operatorname{alg} \mathscr{N} = \{a \in \mathscr{B}(H) \mid pap = ap,\ p \in \mathscr{N}\}.$$

If $\mathscr{K} \cap B(H)$ is the set of compact operators on H let $\mathscr{K}(\mathscr{A}) = \mathscr{A} \cap \mathscr{K}$. It is known [**5**] that $\mathscr{K}(\mathscr{A})$ is ultraweakly dense in $\mathscr{A}$. A simple consequence of this and the duality between compact operators, trace class operators and $B(H)$ is the following.

LEMMA 2. $\mathscr{K}(\mathscr{A})^{**} = \mathscr{A}$ *(where* $\mathscr{K}(\mathscr{A})^{**}$ *denotes the bidual of* $\mathscr{K}(\mathscr{A})$*).*

We state now the analogue of [**1**, Theorem 3.1] for derivations of nest algebras.

THEOREM 3. *Let* δ *be a derivation of* $\mathscr{A}$. *The following conditions are equivalent*:

(1) δ *is weakly compact.*

(2) *The range of* δ *is contained in* $\mathscr{K}(\mathscr{A})$.

(3) $\delta = \operatorname{ad}(k)$ *with* $k \in \mathscr{K}(\mathscr{A})$.

PROOF. (1) $\Rightarrow$ (2) By [**2**] δ is inner, and then $\delta(\mathscr{K}(\mathscr{A})) \subset \mathscr{K}(\mathscr{A})$. By Lemma 2, $\mathscr{K}(\mathscr{A})^{**} = \mathscr{A}$ whence $\delta = (\delta|_{\mathscr{K}(\mathscr{A})})^{**}$. Since δ is weakly compact, $\delta|_{\mathscr{K}(\mathscr{A})}$ is too, so by [**4**, Theorem VI.4.2] it follows that $\delta(\mathscr{A}) \subset \mathscr{K}(\mathscr{A})$.

(2) $\Rightarrow$ (3) follows immediately from [**3**, Theorem 1.4 and **2**].

(3) $\Rightarrow$ (1) follows from the same implication of [**1**, Theorem 3.3].

The preceding result gives necessary and sufficient conditions for a derivation of a nest algebra to be weakly compact. Further, (Theorem 8) we give necessary and sufficient conditions for a nest algebra to admit a nonzero compact derivation.

We need the following well-known (and easy to prove) result.

LEMMA 4. *Let* $\mathscr{A}$ *be a nest algebra and let* $\mathscr{N} = \operatorname{Lat} \mathscr{A}$. *If* $p \in \mathscr{N}$ *then*

$$pB(X)(1-p) \subset \mathscr{A}.$$

LEMMA 5. *If* $\wedge\{p \mid p \in \mathscr{N},\ p \neq 0\} = 0$ *then* $\mathscr{A}$ *does not admit nonzero compact derivations.*

PROOF. The condition of the lemma implies $\dim pH = \infty$ for all $p \in \mathscr{N}$, $p \neq 0$. Let δ be a compact derivation of δ. By Theorem 3, $\delta = \operatorname{ad}(k)$ for some $k \in \mathscr{K}(\mathscr{A})$. We consider the following two possibilities:

(a) There is $p_0 \in \mathscr{N}$, and $\xi_0 \in (1-p_0)H$, $\|\xi_0\| = 1$ such that $\langle k\xi_0, \xi_0\rangle \neq 0$ (here $\langle\ ,\ \rangle$ denotes the inner product in H). Let $k\xi_0 = \lambda\xi_0 + \eta_0$, where $\langle \xi_0, \eta_0\rangle = 0$ and $\lambda \neq 0$. Since $\dim p_0 H = \infty$, let $\{\zeta_n\}_{n=1}^{\infty}$ be an orthonormal family of that space. We consider the operators u_n: $(1-p_0)H \to p_0 H$ defined by

$$u_n\xi_0 = \xi_n, \quad u_n[\xi_0]^{\perp} = 0, \qquad n = 1, 2, \ldots.$$

By Lemma 4, $u_n \in \mathscr{A}$, $n = 1, 2, \ldots$. Since $\|u_n\| = 1$ for all n and δ is compact, the sequence $\{\delta(u_n)\}$ contains a (norm) convergent subsequence which will be denoted by $\{\delta(u_n)\}$ too. On the other hand,

$$\delta(u_n)\xi_0 = ku_n\xi_0 - u_nk\xi_0 = k\zeta_n - \lambda\zeta_n.$$

Since k is compact and $\lambda \neq 0$, this sequence does not contain any convergent subsequence. This contradiction shows that in this case k and so δ must be equal to zero.

(b) $\langle k\xi, \xi\rangle = 0$ for all $\xi \in (1-p)H$ and all $p \in \mathcal{N}$, $p \neq 0$. In this case $(1-p)k = 0$ for all $p \in \mathcal{N}$ and since $\wedge\{p \in \mathcal{N} \mid p \neq 0\} = 0$, it follows that $k = 0$ so $\delta = 0$.

We denote $\mathcal{A}^* = \{a^* \in B(H) \mid a \in \mathcal{A}\}$. Then $\mathcal{A}^*$ is a nest algebra, and $\operatorname{Lat}\mathcal{A}^* = \mathcal{N}^c = \{1 - p \mid p \in N\}$.

LEMMA 6. $\operatorname{ad}(k)$, $k \in \mathcal{K}(\mathcal{A})$, *is a compact derivation of* $\mathcal{A}$ *if and only if* $\operatorname{ad}(k^*)$ *is a compact derivation of* $\mathcal{A}^*$.

PROOF. The proof is obvious.

LEMMA 7. *If* $\wedge\{1 - p \mid p \in \mathcal{N},\ p \neq 1\} = 0$ *then* $\mathcal{A}$ *does not admit nonzero compact derivations.*

PROOF. The proof follows from Lemma 5 applied to $\mathcal{A}^*$, and Lemma 6.

LEMMA 8. *Let* $\delta = \operatorname{ad}(k)$, $k \in \mathcal{K}(\mathcal{A})$, *be a compact derivation of a nest algebra* $\mathcal{A}$. *Then we have*

(a) *If* $q_0 \in \mathcal{N}$ $(= \operatorname{Lat}\mathcal{A})$ *is such that* $\dim(1 - q_0)H = \infty$ *then* $kq_0 = 0$.

(b) *If* $p_0 \in \mathcal{N}$ *is such that* $\dim p_0H = \infty$, *then* $(1 - p_0)k = 0$.

PROOF. (1) Suppose $kq_0 \neq 0$. Then there is $\xi_0 \in q_0H$ with $\|\xi_0\| = 1$, $\zeta_0 \neq 0$, such that $k\xi_0 = \zeta_0$. Since $\dim(1 - q_0)H = \infty$, there is an infinite orthonormal family $\{\zeta_n\}_{n=1}^{\infty}$ of that space. Consider the following family $\{u_n\}$ of operators, u_n: $(1 - q_0)H \to q_0H$:

$$u_n\zeta_n = \xi_0, \quad u_n[\zeta_n]^{\perp} = 0, \qquad n = 1, 2, \ldots.$$

Obviously, $\|u_n\| = 1$, $n = 1, 2, \ldots$. By Lemma 4 $u_n \in \mathcal{A}$ for all n. Since $\operatorname{ad}(k)$ is compact, the sequence $\{ku_n - u_nk\}_{n=1}^{\infty}$ contains a (norm) convergent subsequence which will be denoted by $\{ku_n - u_nk\}_{n=1}^{\infty}$ too. Therefore

$$\lim_{m \geqslant n \to \infty} \|k(u_n - u_m) - (u_n - u_m)k\| = 0.$$

In particular

$$\lim_{m \geqslant n \to \infty} \|k(u_n - u_m)\zeta_n - (u_n - u_m)k\zeta_n\| = 0.$$

Since k is compact and $\{\zeta_n\}$ is an orthonormal family, we have $\lim_{n\to\infty}\|k\zeta_n\| = 0$. Since $\|u_n\| = 1$ for all n we also have $\lim_{m \geqslant n \to \infty}\|(u_n - u_m)k\zeta_n\| = 0$, so

$$\lim_{m \geqslant n \to \infty} \|k(u_n - u_m)\zeta_n\| = 0.$$

But $k(u_n - u_m)\zeta_n = ku_n\zeta_n = k\xi_0 = \zeta_0 \neq 0$. This contradiction proves the lemma. To prove (2) we apply the preceding argument to $\mathcal{A}^*$ and $1 - p_0$ (instead of $\mathcal{A}$ and q_0). We now state our necessary and sufficient conditions for $\mathcal{A}$ to admit a nonzero compact derivation.

THEOREM 9. *Let $\mathscr{A}$ be a nest algebra and $\mathscr{N} = \operatorname{Lat} \mathscr{A}$. The following are equivalent*:

(1) *$\mathscr{A}$ admits a nonzero compact derivation.*

(2) *There is a $p \in \mathscr{N}$, $p \neq 0$, with $\dim pH < \infty$ and a $q \in \mathscr{N}$, $q \neq 1$, such that $\dim(1 - q)H < \infty$.*

PROOF. (2) $\Rightarrow$ (1) Indeed let $k \in pB(H)(1 - q) \subset \mathscr{A}$. Then, it is easy to see that $\operatorname{ad}(k)$ is a compact (actually a finite rank) derivation of $\mathscr{A}$.

(1) $\Rightarrow$ (2) Suppose first that $\dim pH = \infty$ for all $p \in \mathscr{N}$, $p \neq 0$. Let $p_0 = \wedge\{p \in \mathscr{N} \mid p \neq 0\}$. Then, either $p_0 = 0$ or $\dim p_0 H = \infty$. If $p_0 = 0$ then by Lemma 5, $\mathscr{A}$ does not admit nonzero compact derivations. Assume $p_0 \neq 0$ and let δ be a compact derivation of $\mathscr{A}$. By Theorem 3, $\delta = \operatorname{ad}(k)$ for some $k \in \mathscr{K}(\mathscr{A})$. Since $\mathscr{A}$ is a nest algebra, it follows that $p_0 \mathscr{A} p_0 = B(p_0 H)$. Since kp_0 implements a compact derivation of $p_0 \mathscr{A} p_0 = B(p_0 H)$ and $\dim p_0 H = \infty$, by Lemma 1 we have $kp_0 = p_0 k p_0 = 0$. On the other hand, by Lemma 8, we have $(1 - p_0)k = 0$. Therefore $k = p_0 k(1 - p_0)$. It remains to prove that $p_0 k(1 - p_0) = 0$. Assume the contrary. Then there is $\zeta \in (1 - p_0)H$, $\zeta \neq 0$, with $k\zeta = \xi_0 \in p_0 H$, $\|\xi_0\| = 1$. Since $\dim p_0 H = \infty$, let $\{\zeta_n\}_{n=1}^{\infty}$ be an orthonormal family of that infinite dimensional space. Let $\{u_n\}_{n=1}^{\infty} \subset B(p_0 H) = p_0 \mathscr{A} p_0 \subset \mathscr{A}$, $\|u_n\| = 1$ be the family of operators defined by

$$u_n \xi_0 = \zeta_n, \quad u_n[\xi_0]^{\perp} = 0, \qquad n = 1, 2, \ldots.$$

Then, since $\operatorname{ad}(k)$ is a compact derivation of $\mathscr{A}$, the sequence $\{(ku_n - u_n k)\zeta\}_{n=1}^{\infty}$ contains a norm convergent subsequence. But $(ku_n - u_n k)\zeta = -u_n k\zeta = -\zeta_n$ so this sequence does not contain any convergent subsequence. This contradiction shows that $p_0 k(1 - p_0) = 0$ and hence $k = 0$. Assume now that $\dim(1 - q)H = \infty$ for all $q \in \mathscr{N}$, $q \neq 1$. Then, the preceding argument applied to $\mathscr{A}^*$ shows that $\mathscr{A}^*$ does not admit nonzero compact derivations. By Lemma 6, $\mathscr{A}$ does not admit nonzero compact derivations and the proof of Theorem 9 is complete.

THEOREM 10. *Every compact derivation of a nest algebra $\mathscr{A}$ is the norm limit of finite rank derivations.*

PROOF. Let

$$p_0 = \bigvee \{p \in \mathscr{N} \mid \dim pH < \infty\} \quad \text{and} \quad q_0 = \bigwedge \{q \in \mathscr{N} \mid \dim(1 - q)H < \infty\}.$$

Let also $\delta = \operatorname{ad}(k)$, $k \in \mathscr{K}(\mathscr{A})$, be a compact derivation of $\mathscr{A}$. We first prove that $k = p_0 k(1 - q_0)$. We note that $(1 - p_0)k$ implements a compact derivation of the nest algebra $(1 - p_0)\mathscr{A}(1 - p_0)$. If $\dim p_0 < \infty$ then $\operatorname{Lat}(1 - p_0)\mathscr{A}(1 - p_0)$ does not contain any finite dimensional projection, so by Theorem 9, $(1 - p_0)k = 0$. If $\dim p_0 = \infty$ we have $(1 - p_0)k = 0$ by Lemma 8(b). Hence $k = p_0 k$. A similar argument shows that $k = k(1 - q_0)$. From the definition of p_0 (respectively q_0) it follows that there exists a sequence $\{p_n\} \subset \mathscr{N}$, $\dim p_n H < \infty$ (respectively $\{q_n\} \subset \mathscr{N}$, $\dim(1 - q_n)H < \infty$) such that $p_n \uparrow p_0$ (respectively $q_n \downarrow q_0$); of course, if $\dim p_0 H < \infty$ (respectively $\dim(1 - q_0)H < \infty$), the sequence $\{p_n\}$ (respectively

$\{q_n\}$) will be constant. Since k is compact, it follows that

$$k = p_0 k(1 - q_0) = (\text{norm}) \lim_{n \to \infty} p_n k(1 - q_n).$$

As obviously $\mathrm{ad}(p_n k(1 - q_n))$ is finite rank the proof of Theorem 10 is complete.

References

1. Ch. Akemann and S. Wright, *Compact and weakly compact derivations of C*-algebras*, Pacific J. Math. **85** (1979), 253–259.

2. E. Christensen, *Derivations of nest algebras*, Math. Ann. **229** (1977), 155–161.

3. E. Christensen and C. Peligrad, *Commutants of nest algebras module the compact operators*, Invent. Math. **56** (1980), 113–116.

4. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.

5. J. A. Erdos, *Operators of finite rank in nest algebras*, J. London Math. Soc. **43** (1968), 391–397.

6. Y. Ho, *A note on derivations*, Bull. Inst. Math. Acad. Sinica **5** (1977).

7. B. E. Johnson and S. K. Parrott, *Operators commuting with a von Neumann algebra modulo the set of compact operators*, J. Funct. Anal. **11** (1972), 39–61.

8. S. Popa, *The commutant modulo the set of compact operators of a von Neumann algebra*, preprint, Increst, 1985.

9. S. K. Tsui and S. Wright, *Asymptotic commutants and zeroes of von Neumann algebras*, Math. Scand. **51** (1982), 232–240.

10. S. Wright, *Banach-module-valued derivations on C*-algebras*, Illinois J. Math. **24** (1980), 462–467.

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