PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 4, August 1986

DISCRIMINATORY STABLE SETS FOR (n, k) GAMES

CHIH CHANG[1]

ABSTRACT. The description of the explicit form of particular vN-M stable sets for special classes of n-person cooperative games is interesting and important from the viewpoint of appliction as well as theory. All discriminatory vN-M stable sets for (n, k) games which were defined by Muto are characterized.

The von Neumann-Morgenstern solution for n-person games usually characterizes many different stable sets for most games. von Neumann and Morgenstern state that, among the several stable sets, the ones which are accepted depend upon "standards of behavior" which are norms, rules or conventions imposed by society. In studying stable sets for n-person games, we can often see that one, or several, of the players always receives the same amount. Such players are said to be *discriminated players*, and such stable sets are called discriminatory. This concept was introduced by von Neumann and Morgenstern [7, p. 290]. It is of interest to examine games in which the social standard of discrimination is involved.

The purpose of this paper is to characterize all discriminatory stable sets for any (n, k) game.

1. Basic defintiions and theorem. An *n-person game* is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a set of n *players* and v is a real-valued *characteristic function* on 2^N with $v(\varnothing) = 0$. Here 2^N denotes the set of all subsets of N and $\varnothing$ is the empty set. Any nonempty subset of N will be called a *coalition*.

A game (N, v) is said to be (0, 1)-*normalized* if $v(N) = 1$ and $v(\{i\}) = 0$ for all $i \in N$. Most games (N, v) can be converted to their (0, 1)-normalized form without changing their essential structure, nor the basic nature of most solution concepts. So we will assume (0, 1)-normalized games throughout. A game (N, v) is said to be a *constant sum* game if $v(S) + v(N - S) = v(N)$, where $S \subseteq N$.

The set of *imputations* is

$$A = \left\{ x \in R^n \colon \sum_{i \in N} x_i = 1 \text{ and } x_i \geqslant 0 \text{ for all } i \in N \right\}.$$

For any x and $y \in A$ and nonempty $S \subseteq N$, we say x *dominates* y via S, denoted by $x \operatorname{dom} y$ via S or $x \operatorname{dom}_S y$, if $x_i > y_i$ for all $i \in S$ and $\sum_{i \in S} x_i \leqslant v(S)$. This latter inequality is referred to as S and is *coalition effective* for x. We also say x

Received by the editors March 25, 1985, and, in revised form, July 16, 1985.
1980 *Mathematics Subject Classification*. Primary 90D12.
Key words and phrases. (n, k) games, discriminatory stable set.
[1] The author wishes to acknowledge Professor W. F. Lucas of Claremont Graduate School for his many helpful suggestions.

dominates y, denoted by $x \operatorname{dom} y$, if there is some S such that $x \operatorname{dom} y$ via S. For any $B \subseteq A$, we define

$$\operatorname{Dom}_S B = \{ y \in A\colon x \operatorname{dom} y \text{ via } S \text{ for some } x \in B \},$$
$$\operatorname{Dom} B = \bigcup_{S \subset N} \operatorname{Dom}_S B.$$

A subset V of A is said to be a *vN-M stable set* or *stable set* if and only if $V \cap \operatorname{Dom} V = \varnothing$ and $V \cup \operatorname{Dom} V = A$. These two conditions are called *internal* and *external* stability, respectively. We say that a coalition S is vital if $v(S) > 0$.

A game (N, v) is said to be a *symmetric game* if $V(S) = v(T)$ whenever $|S| = |T|$, i.e., whenever S and T contain the same number of players. In this case, we also write $v(S) = v(s)$, where $|S| = s$ is the cardinality of S. We say that a symmetric game (N, v) *has strongly vital k-person coalitions* if $v(s) \leqslant v(k) \cdot (s/k)$ for all $k \leqslant s < n$ and $v(s) = 0$ for all $s < k$. The symbol (n, k) will be used to denote such games.

To simply the notation let $y(S) = \sum_{i \in S} y_i$ whenever $y \in A$ and the nonempty set $S \subseteq N$.

The following theorem was proved by Hart [4]:

THEOREM 1.1. *Consider any* (n, k) *game. For any* $x,\ y \in A$ *and any* $T \subseteq N$, *if* $x \operatorname{dom} y$ *via* T *then there is some* $S \subseteq N$ *such that* $|S| = k$ *and* $x \operatorname{dom} y$ *via* S.

DEFINITION. A *p-discriminatory set*, where p is a positive integer, is a set $V(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_p}; i_1, i_2, \ldots, i_p) = \{x \in A\colon x_{i_k} = \alpha_{i_k}, \text{ where } i_k \in N \text{ and } 1 \leqslant k \leqslant p\}$.

V_p will denote a p-discriminatory set, and $D_p = \{i_1, i_2, \ldots, i_p\}$, i.e., the set of discriminated players in V_p. If a p-discriminatory set V_p is a stable set for the game (N, v), then we will call it a *p-discriminatory stable set* for (N, v).

Discriminatory stable sets have been determined for several classes of games. Herbert [3] descrsibed the 2-discriminatory stable set for 4-person constant sum games. A necessary and sufficient condition for games to have $(n-2)$-discriminatory stable sets with discriminated players $\{3, 4, \ldots, n\}$ and $\alpha_3 \geqslant \alpha_4 \geqslant \cdots \geqslant \alpha_n$ was obtained by Owen [5]. Owen [6] has also characterized all occurrences of discrimination for games in which only the 1, $(n-1)$, and n-player coalitions are vital. Weber [9, 10] extended these results to games in which the $(n-2)$-player coalitions may also be vital and characterized all discriminatory stable sets for (n, k) games with $v(k) = 1$ and $k > n/2$.

2. Main theorems. We will now describe discriminatory stable sets for any (n, k) game.

In Theorems 2.1 and 2.2, we will show that if V_l is a discriminatory stable set for a game (n, k) with $v(k) = \alpha$, then $l = n - k$ and $x(N - D_{n-k}) \leqslant \alpha$ for each $x \in V_l$.

THEOREM 2.1. *If* V_l *is a discriminatory stable set for an* (n, k) *game with* $v(k) = \alpha$, *then* $l = n - k$.

PROOF. Let V_l be any l-discriminatory set. The proof will be carried out by reductio ad absurdum.

Case 1. $l > n - k$. There are two subcases.

(i) Assume $\sum_{h=1}^{l} \alpha_{i_h} < 1$. Define x by

$$x_i = \begin{cases} \alpha_{i_h} + \varepsilon/l & \text{for } i_h \in D_l, \\ 0 & \text{otherwise} \end{cases}$$

where $\varepsilon = 1 - \sum_{h=1}^{l} \alpha_{i_h}$. Then $x \notin V_l$.

For any $y \in V_l$, there are at most $n - l < n - (n - k) = k$ components larger than the corresponding components of x. Therefore, $x \notin \operatorname{Dom} V_l$ and V_l is not a stable set.

(ii) Assume $\sum_{h=1}^{l} \alpha_{i_h} = 1$. V_l is a one-point set. There exists at least one $k \in \{1, 2, \ldots, l\}$ such that $\alpha_{i_k} > 0$. Define z by

$$z_i = \begin{cases} z_i + \left[\alpha_{i_k}/(n-1)\right] & \text{for } i \in N - \{i_k\}, \\ 0 & \text{for } i = i_k. \end{cases}$$

Then, $z \in A$ and V_l cannot dominate z. Hence, V_l is not a stable set.

Case 2. $l < n - k$. $n - l > n - (n - k) = k$. We can choose an $S \subset N - D_l$ with $|S| = k$. There exists an $x \in V_l$ such that $x(S) \leqslant v(S) = v(k)$ and $x_i > 0$ for each $i \in S$. Define y by

$$y_i = \begin{cases} \alpha_i & \text{for } i \in D_l, \\ x_i - \varepsilon_i & \text{for } i \in S, \\ x_i + \left(\sum_{j \in S} \varepsilon_j \right) / |S| & \text{for } i \in N - (S \cup D_l) \end{cases}$$

where $\varepsilon_i > 0$ is small enough such that $x_i - \varepsilon_i \geqslant 0$ for each $i \in S$. Then $y \in Y_l$ and $x \operatorname{dom} y$. Therefore, V_l cannot satisfy internal stability. □

For an $(n, n-1)$ game with $v(n-1) = 1$, von Neumann and Morgenstern **[8]** showed that the minimum payoff to the nth player is always less than $1/(n-1)$ for stable sets which are symmetric in just the first $n - 1$ players. In Theorem 2.2, we will show that the minimum payoff to all discriminated players is no less than $1 - \alpha$ for an (n, k) game with $v(k) = \alpha$.

THEOREM 2.2. *If V_{n-k} is a discriminatory set for an (n, k) game with $v(k) = \alpha$, then $x(N - D_{n-k}) \leqslant \alpha$ for any $x \in V_{n-k}$.*

PROOF. Suppose to the contrary that $x(N - D_{n-k}) > \alpha$. Then $1 - \alpha > x(D_{n-k}) \geqslant 0$. Let $Y = \{y \in A \mid y_i = \alpha_i + \varepsilon \text{ for } i \in D_{n-k}, \text{ where } \varepsilon > 0\}$. It is easy to see that $Y \cap V_{n-k} = \varnothing$. Then each y in Y must be dominated by at least one point x in V_{n-k}. From the definition of Y, we can see that y can be dominated by x only via $N - D_{n-k}$. However, $x(N - D_{n-k}) > \alpha$ for each $x \in V_{n-k}$. So, V_{n-k} does not satisfy external stability. □

Next, we describe a necessary and sufficient condition for a set V_{n-k} with $x(N - D_{n-k}) \leqslant \alpha$ for all $x \in V_{n-k}$ to be a stable set for an (n, k) game with

$v(k) = \alpha$. In order to do this, we will first introduce a lemma which will be useful later. Consider the inequality system:

$$\text{I} \quad \begin{cases} A_i x \geqslant a_i & \text{for } i = 1, 2, \ldots, m, \\ A_0 x = a_0 \end{cases}$$

where the $A_i \neq 0$, for each $i = 0, 1, 2, \ldots, m$, are vectors in R^n with nonnegative components, $x \in R^n$ and the a_i, for $i = 0, 1, 2, \ldots, m$, are nonnegative real numbers.

DEFINITION. A vector $b = (b_1, b_2, \ldots, b_m)$, where $b_i \geqslant 0$ for each $i = 1, 2, \ldots, m$, which satisfies $\sum_{i=1}^m b_i A_i = A_0$ is called a *cover of* the vector A_0 *by means of* $A_1, A_2, \ldots, A_m$. *A* cover b is said to be *reduced* if the vectors $\{A_i \mid b_i > 0$, where $\sum_{i=1}^m b_i A_i = A_0\}$ are linearly independent.

The following lemma which was derived by Bondareva [**1**, **2**] states a necessary and sufficient condition for the inequality system I to possess a solution.

LEMMA 2.3. *The system* I *possesses a solution if and only if the condition*

$$\sum_{i=1}^{m} b_i a_i \leqslant a_0 \tag{2.1}$$

is satisfied for any reduced cover $b = (b_1, b_2, \ldots, b_m)$. *Moreover, if* $A_{i_0} x > a_{i0}$ *and* $b_{i_0} > 0$ *for* $i_0 \in \{1, 2, \ldots, m\}$, *then* $\sum_{i=1}^m b_i a_i < a_0$.

THEOREM 2.4. *The* $(n - k)$*-discriminatory set* V_{n-k} *with* $\bar{\alpha} = \sum_{i \in D_{n-k}} \alpha_i \geqslant 1 - \alpha$ *is a stable set for any* (n, k) *game with* $v(k) = \alpha$ *if and only if* $\alpha > [(k-1)/k] + \alpha_M$, *where* $\alpha_M = \max\{\alpha_i : i \in D_{n-k}\}$.

PROOF. For convenience, let $S = N - D_{n-k}$. First, we will prove the internal stability of V_{n-k}.

Take any $x, y \in V_{n-k}$. Then $x_l = y_l$ for each $l \in D_{n-k}$ and $x(S) = y(S) = 1 - \bar{\alpha}$. Hence, x cannot dominate y and y cannot dominate x.

To prove external stability, pick any $y \in A - V_{n-k}$.

Case 1. $y(S) < 1 - \bar{\alpha}$. Let $\varepsilon = 1 - \bar{\alpha} - y(S) > 1 - \bar{\alpha} - (1 - \bar{\alpha}) = 0$. Define x by

$$x_i = \begin{cases} y_i + \varepsilon/k & \text{for } i \in S, \\ \alpha_i & \text{for } i \in D_{n-k}. \end{cases}$$

Then $x \in V_{n-k}$ and $x \operatorname{dom} y$.

Case 2. $y(S) \geqslant 1 - \bar{\alpha}$. Since $y(D_{n-k}) = 1 - y(S) \leqslant 1 - (1 - \bar{\alpha}) = \bar{\alpha}$ there exists an $i \in D_{n-k}$ such that $y_i < \alpha_i$. Otherwise, $y_i \geqslant \alpha_i$, for each $i \in D_{n-k}$, which implies $y_i = \alpha_i$ for each $i \in D_{n-k}$. This violates the assumption that $y \in A - V_{n-k}$. Let $h \in D_{n-k}$ be such that $y_h < \alpha_h$.

We will describe a sufficient condition for the existence of an $x \in V_{n-k}$ such that $x \operatorname{dom} y$.

If any one of the following k inequality systems I_j, $j \in S$, holds, then $x \operatorname{dom} y$.

$$\mathrm{I}_j \quad \begin{cases} x_i > y_i & \text{for each } i \in S - \{j\}, \\ \alpha_h < y_h & \text{for some } h \in D_{n-k}, \\ \sum_{i \in S - \{j\}} x_i + \alpha_h \leqslant \alpha, & \\ \sum_{i \in S} x_i = 1 - \bar{\alpha} & \end{cases}$$

where $j \in S$.

We can see that $\sum_{i \in S - \{j\}} x_i + \alpha_h \leqslant \alpha$ is equivalent to $x_j \geqslant 1 - \alpha - \sum_{l \in D_{n-k} - \{h\}} \alpha_l$. Therefore, the only reduced cover of the system can be observed to be $(1, 1, \ldots, 1)$. Using Lemma 2.3, we know that the system I_j is solvable if and only if $\sum_{i \in S - \{j\}} y_i + 1 - \alpha - \sum_{l \in D_{n-k} - \{h\}} < 1 - \bar{\alpha}$. We need to have at least one of these k inequality systems hold. That is to say, there exists no $y \in A - V_{n-k}$ with $y(S) \geqslant 1 - \bar{\alpha}$ such that the following system holds:

$$\mathrm{II} \quad \begin{cases} \sum_{i \in S - \{j\}} y_i + 1 - \alpha - \sum_{l \in D_{n-k} - \{h\}} \alpha_l \geqslant 1 - \bar{\alpha} & \text{for each } j \in S, \\ y_i \geqslant 0 & \text{for each } i \in N, \\ \sum_{i \in S} y_i \geqslant 1 - \bar{\alpha}, & \\ \sum_{i \in N} y_i = 1. & \end{cases}$$

We can also see that the first inequality of system II is equivalent to $\sum_{i \in S - \{j\}} y_i \geqslant \alpha - \alpha_h$ for each $j \in S$.

We can change system II to II′:

$$\mathrm{II}' \quad \begin{cases} I_{S - \{j\}} y \geqslant \alpha - \alpha_h & \text{for each } j \in S, \\ I_{\{i\}} y \geqslant 0 & \text{for each } i \in N, \\ I_S y \geqslant 1 - \bar{\alpha}, & \\ I_N y = 1 & \end{cases}$$

where $I_W = (c_1, c_2, \ldots, c_n) \in R^n$, $W \subset N$, $c_i = 1$ if $i \in W$, and $c_i = 0$, if $i \in N - W$.

There must exist a reduced cover for which condition (2.1) in Lemma 2.3 fails, for otherwise, system II′ has a solution. We shall list all reduced covers for system II′.

Let b_W denote the component of the reduced cover corresponding to I_W.

Subcase 1. $b_S = c$, where c is a constant and $0 < c \leqslant 1$.

(i) Assume $b_S = 1$ and $b_{\{i\}} = 1$ for each $i \in D_{n-k}$. Then $b_S(1 - \bar{\alpha}) + \sum_{i \in D_{n-k}} b_{\{i\}} \cdot 0 = 1 - \bar{\alpha} < 1$. So, this reduced cover does satisfy condition (2.1) in Lemma 2.3.

(ii) $0 < b_S < 1$. There does not exist any reduced cover with $b_S < 1$, for otherwise, there must exist a $b_{S - \{j\}}$ and a $b_{\{i\}}$, where $i, j \in S$, such that $\sum_{j \in S} b_{S - \{j\}} I_{S - \{j\}} + \sum_{i \in S} b_{\{i\}} I_{\{i\}} + b_S I_S = I_S$. This implies $I_S = [1/(1 - b_S)](\sum_{j \in S} b_{S - \{j\}} I_{S - \{j\}} + \sum_{i \in S} b_{\{i\}} I_{\{i\}})$. The vectors $\{I_W : b_W > 0\}$ are linearly dependent.

Subcase 2. $b_S = 0$. Set $b_{S-\{j\}} = 1/(k-1)$ and $b_{\{i\}} = 1$ for all $i \in N - S$. Then, $[1/(k-1)]\Sigma_{j \in S}(\alpha - \alpha_h) = [k/(k-1)](\alpha - \alpha_h) \geqslant 1$ if and only if $\alpha \geqslant \alpha_h + (k-1)/k$.

So far, we have obtained a necessary and sufficient condition of system I to have a solution. In order to show that V_{n-k} dominates any $y \in A - V_{n-k}$, we need to consider each $h \in D_{n-k}$. Therefore, a sufficient condition for V_{n-k} to dominate any $y \in A - V_{n-k}$ is that $\alpha \geqslant [(k-1)/k] + \alpha_M$.

It remains to prove that $\alpha \geqslant [(k-1)/k] + \alpha_M$ is also a necessary condition.

Let $A_M = \{y \in A - V_{n-k} : y_M < \alpha_M$ and $y_l \geqslant \alpha_l$ for each $l \in D_{n-k} - \{M\}$. It is easy to see that $A_M \neq \varnothing$. Using the same procedure as in Case 2 of this theorem, we can prove that a necessary and sufficient condition for $A_M \operatorname{Dom} V_{n-k}$ is $\alpha \geqslant [(k-1)/k] + \alpha_M$. Hence, the range of α cannot be extended. We have the desired result. □

Thus, we have described all discriminatory stable sets for any (n, k) game.

Next, we shall give two examples of a discriminatory stable set for (n, k) games.

EXAMPLE 1. *All discriminatory stable sets for any* (3, 2) *game*. If any (3, 2) game does have discriminatory stable sets, they must be 1-discriminatory stable sets. W.L.O.G. we assume that player 1 is discriminated. Using Theorem 2.4, we know that any (3, 2) game with $v(2) = a = 3/4 + \varepsilon$, where $1/4 \geqslant \varepsilon > 0$, has 1-discriminatory stable sets. $V_{y_1} = \{(y_1, x_2, x_3) \in A \mid x_2 + x_3 = 1 - y_1\}$, where $y \in [1/4 - \varepsilon, 1/4 + \varepsilon)$ (see Figure 2.1). The triangle represents the set A and the heavy line indicates V_{y_1}.

EXAMPLE 2. *All discriminatory stable sets for the* (4, 2) *game with* $v(2) = 0.8$. We know that only 2-discriminatory sets can be discriminatory stable sets for any (4, 2) game. W.L.O.G. we assume that players 1 and 2 are discriminated and player 1 gets no less than player 2. Then, $V_{y_1, y_2} = \{(y_1, y_2, x_3, x_4) \in A : x_3 + x_4 = 1 - y_1 - y_2\}$, where $y_1 \geqslant y_2$, $0.3 > y_1 \geqslant 0.1$, and $y_1 + y_2 \geqslant 0.2$ are discriminatory stable sets for this game.

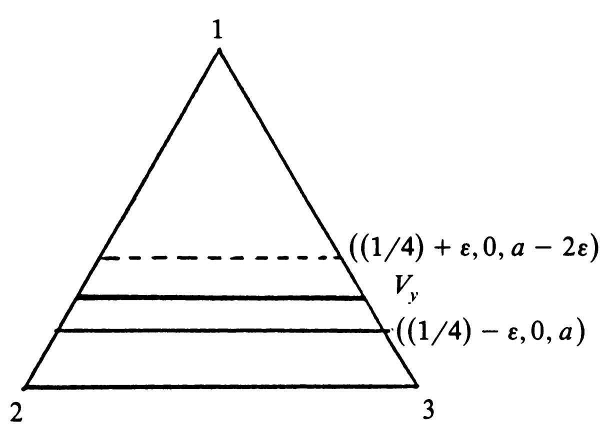

FIGURE 2.1

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INSTITUTE OF APPLIED MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, HSINCHU 30043, TAIWAN, REPUBLIC OF CHINA