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SELECTION THEOREMS AND INVARIANCE OF BOREL POINTCLASSES

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ABSTRACT. We generalize some known selection theorems and give simple proofs of results on the invariance of Borel pointclasses obtained by Saint-Raymond, Jayne and Rogers, and Kunen and Miller.

1. Introduction. In [**9**], Saint-Raymond proved the following selection theorem.

THEOREM 1.1. *Suppose X and Y are compact metrizable spaces and Z is a second countable metrizable space. If $f\colon X \to Y$ is a continuous surjection and $p\colon X \to Z$ is a Borel measurable function of class α, then there is a class* 1 *map $s\colon Y \to X$ such that $p \circ s$ is of class α and $f(s(y)) = y$ for all y.*

Jayne and Rogers [**3**, **4**] have made a detailed study of this result and have extended it to the nonseparable case for continuous as well as for class 1 maps. Here we extend the result of Saint-Raymond in the more general set-up of Kuratowski and Ryll-Nardzewski [**7**] and Debs [**1**]. Our proof is simpler than those of Saint-Raymond, Jayne and Rogers.

As an application Saint-Raymond, Jayne and Rogers gave results on the complexity of preimages of Borel sets. For the sake of completeness we indicate these applications in our paper.

2. Notation and preliminaries. Throughout T will denote an arbitrary set, $\mathfrak{A}$ a family of subsets of T, and X, Y, Z second countable metrizable spaces. A second countable, completely metrizable space is called a *Polish space*. Given $\mathfrak{A}, \mathfrak{A}_\sigma$ ($\mathfrak{A}_\delta$) will denote the family of unions (intersections) of a sequence of sets in $\mathfrak{A}$.

A *multifunction* $F\colon T \to X$ is a map defined on T whose values are nonempty subsets of X. For $E \subseteq X$,

$$F^{-1}(E) = \{t \in T\colon F(t) \cap E \neq \varnothing\}.$$

We say that F is $\mathfrak{A}$-*measurable* (*strongly* $\mathfrak{A}$-*measurable*) if $F^{-1}(E) \in \mathfrak{A}$ for every open (closed) set E in X. In particular, a point map $f\colon T \to X$ is $\mathfrak{A}$-measurable if $f^{-1}(E) \in \mathfrak{A}$ for every open subset E of X. A function $s\colon T \to X$ is called a *selector* for F if $s(t) \in F(t)$ for every $t \in T$. The set

$$\{(t, x) \in T \times X\colon x \in F(t)\}$$

is called the *graph* of F and is denoted by Graph(F). If T is a topological space then $F\colon T \to X$ is called *lower semicontinuous* (*upper semicontinuous*) if $F^{-1}(A)$ is open (closed) for every open (closed) set A in X.

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For notation and terminology in descriptive set theory we follow Kuratowski [**6**]. The set of natural numbers will be denoted by ω. Further, $\omega^{<\omega} = \bigcup_{k\in\omega} \omega^k$. We shall use e to denote the empty sequence. A map $f\colon X \to Y$ is called an *open* (*closed*) *map* if $f(X) = Y$ and the image of every open (closed) set in X is open (closed) in Y. For $A \subseteq X$, $\mathrm{cl}(A)$ will denote the closure of A in X.

3. Main results. From now on, $\mathcal{L}$ is a field of subsets of T, X and Y are Polish spaces and Z is a second countable metrizable space.

THEOREM 3.1. *If $F\colon T \to X$ is a closed valued, strongly $\mathcal{L}_\sigma$-measurable multifunction and $g\colon X \to Z$ is a class 1 map, then there is an $\mathcal{L}_\sigma$-measurable selector $s\colon T \to X$ for F such that $g \circ s$ is also $\mathcal{L}_\sigma$-measurable.*

PROOF. Fix a complete metric d on X compatible with its topology such that d-diameter$(X) < 1$.

Step 1. There are systems of subsets $\{T(s)\colon s \in \omega^{<\omega}\}$ and $\{H(s)\colon s \in \omega^{<\omega}\}$ of T and X respectively such that, for every $s \in \omega^k$ and $n, m \in \omega$ $(n \neq m)$,

(i) $T(s) \in \mathcal{L}_\sigma, T(e) = T$,

(ii) $T(sn) \cap T(sm) = \varnothing$,

(iii) $T(s) = \bigcup_{j\in\omega} T(sj)$,

(iv) $H(s)$ is closed and of d-diameter $< 2^{-k}$,

(v) d'-diameter$(g(H(s))) < 1/2^k$ (where d' is a fixed metric on Z compatible with its topology),

(vii) for every $t \in T(s), F(t) \cap H(s) \neq \varnothing$.

To see that such systems exist we shall proceed inductively. Define $T(e) = T$ and $H(e) = X$. Suppose for every $s \in \bigcup_{i=0}^{k} \omega^i, T(s)$ and $H(s)$ have been defined satisfying (i)–(vii). Fix an $s \in \omega^k$. Fix a base $W_0, W_1, \ldots$ for Z such that d'-diameter$(W_i) < 2^{-(k+1)}$ for all i. Also, fix a base $V_0, V_1, \ldots$ for X such that, for each i, d-diameter$(V_i) < 2^{-(k+1)}$.

Let $g^{-1}(W_i) = \bigcup_{j\in\omega} E_{ij}$, where E_{ij} are closed in X. Enumerate $\{E_{ij}\colon i \in \omega, j \in \omega\}$ in a single sequence $F_0, F_1, \ldots$. For $m, n \in \omega$, let

$$T'(s)(n,m) = \{t \in T(s)\colon F(t) \cap H(s) \cap \mathrm{cl}(V_m) \cap F_n \neq \varnothing\}.$$

Since F is strongly $\mathcal{L}_\sigma$-measurable and $T(s) \in \mathcal{L}_\sigma, T'(s)(n,m) \in \mathcal{L}_\sigma$, $\forall m, n \in \omega$. Enumerate $\{T'(s)(n,m)\colon m, n \in \omega\}$ in a single sequence $\{T''(s)(n)\colon n \in \omega\}$. By [**7**] we get pairwise disjoint sets $T(sn)$, $n \in \omega$, in $\mathcal{L}_\sigma$ such that

$$\bigcup_{n\in\omega} T(sn) = \bigcup_{n\in\omega} T''(s)(n)$$

and

$$T(sn) \subseteq T''(s)(n) \quad \text{for all } n.$$

Put

$$H(sn) = H(s) \cap \mathrm{cl}(V_j) \cap F_i$$

whenever $T(sn) \subseteq T'(s)(i,j)$. This completes the definition of two systems of sets with required properties.

Step 2. We now define the selector s for F. Fix $t \in T$. There is a unique $\alpha \in \omega^\omega$ such that $t \in T(\alpha|k)$ for all k. By (iv) and (v), $\bigcap_{k\in\omega} H(\alpha|k)$ is a singleton. We define $s(t)$ to be the unique point of $\bigcap_{k\in\omega} H(\alpha|k)$.

To prove that s is $\mathcal{L}_\sigma$-measurable, fix a closed set $E \subseteq X$. Let

$$E_k = \{x \in X\colon (\exists y \in E)(d(y,x) < 1/2^k)\}, \qquad k \in \omega.$$

Also, note that for each k, the multifunction $F_k\colon T \to X$ defined by

$$F_k(t) = F(t) \cap H(n_0, \dots, n_{k-1}) \quad \text{if } t \in T(n_0, \dots, n_{k-1})$$

is strongly $\mathcal{L}_\sigma$-measurable. It is easy to check that

$$s^{-1}(E) = \bigcap_{k\in\omega} \{t \in T\colon F_k(t) \subseteq E_k\}.$$

Hence, $s^{-1}(E) \in \mathcal{L}_\delta$. Thus, s is $\mathcal{L}_\sigma$-measurable.

Now, it only remains to show that $g \circ s$ is $\mathcal{L}_\sigma$-measurable. To see this choose a point $x(n_0, \dots, n_{k-1}) \in H(n_0, \dots, n_{k-1})$ for each $(n_0, \dots, n_{k-1})$. Define $f_k\colon T \to X$ by

$$f_k(t) = x(n_0, n_1, \dots, n_{k-1}) \quad \text{if } t \in T(n_0, \dots, n_{k-1}).$$

It is easy to see that $g \circ f_k$ is $\mathcal{L}_\sigma$-measurable. Further, by (v), $\{g \circ f_k\}_{k\in\omega}$ converges uniformly to $g \circ s$. Hence, $g \circ s$ is $\mathcal{L}_\sigma$-measurable [**7**]. The proof is complete.

We now extend this theorem for G_δ-valued multifunctions.

THEOREM 3.2. *Let $T, X, Z, \mathcal{L}$ and g be as in the previous theorem. Suppose $F\colon T \to X$ is a strongly $\mathcal{L}_\sigma$-measurable multifunction such that*

$$\operatorname{Graph}(F) \in (\mathcal{L} \times \mathcal{U})_{\sigma\delta}$$

where

$$\mathcal{L} \times \mathcal{U} = \{E \times U \subseteq T \times X\colon E \in \mathcal{L} \text{ and } U \subseteq X \text{ is open}\}.$$

Then there is an $\mathcal{L}_\sigma$-measurable selector $s\colon T \to X$ for F such that $g \circ s$ is also $\mathcal{L}_\sigma$-measurable.

PROOF. Set $\operatorname{Graph}(F) = \bigcap_{k=1}^\infty G_k$, where $G_k = \bigcup_{n\in\omega}(E_{nk} \times U_{nk})$, $E_{nk} \in \mathcal{L}$ and U_{nk} open in X, for all n and k.

A slight modification of the argument in Step 1 of the previous theorem gives us a system of sets $\{T(s)\colon s \in \omega^{<\omega}\}$ and $\{H(s)\colon s \in \omega^{<\omega}\}$ in T and X respectively which satisfy conditions (i)–(vii) and

(viii) for every $k \in \omega$, $\bigcup_{s\in\omega^k}\{T(s) \times H(s)\} \subseteq G_k$.

Now we follow the arguments of Theorem 3.1 and obtain a selector s with required properties.

REMARK. The set up of Theorem 3.2 is very similar to that of the main theorem in [**1**].

4. Selection theorems for lower semicontinuous and upper semicontinuous multifunctions. As a corollary to Theorem 3.1, we now give several generalizations to the selection theorem of Saint-Raymond. See also [**3**, **4**].

THEOREM 4.1. *Let X be a Polish space and let T, Z be second countable completely metrizable spaces. Suppose $g\colon X \to Z$ is a Borel measurable function of class α $(1 \leq \alpha < \omega_1)$ and $F\colon T \to X$ is an upper semicontinuous (u.s.c.) closed valued* *multifunction. Then there is a class* 1 *selector* $s\colon T \to X$ *for* F *such that* $g \circ s$ *is of class* α.

PROOF. Since every u.s.c. closed valued multifunction admits a class 1 selector, the result is true for $\alpha > \omega_0$. To prove the result for finite α, we proceed by induction.

Suppose $\alpha = 1$. Take $\mathcal{L}$ to be the family of subsets of T which are simultaneously F_σ and G_δ and apply Theorem 3.1.

Assume the result is true for $\alpha = m$. If g is of class $(m+1)$ then there is a sequence of class m function $g_n\colon X \to Z$ which converges pointwise to g. Define $h\colon X \to Z^\omega$ by

$$h(x) = (g_0(x), g_1(x), \ldots), \qquad x \in X.$$

By induction hypothesis, we get a class 1 selector $s\colon T \to X$ for F such that $h \circ s$ is of class m. Then $g \circ s = \lim_n g_n \circ s$, and hence is of class $(m+1)$.

REMARK. The induction argument above is due to Saint-Raymond.

THEOREM 4.2. *The previous theorem is also true when* F *is lower semicontinuous (l.s.c.).*

PROOF. We need to prove the result for $\alpha = 1$ only. By a result of Michael [**8**] there is a compact valued, u.s.c. multifunction $H\colon T \to X$ such that $H(t) \subseteq F(t)$ for all t. We use Theorem 4.1 for H and g.

THEOREM 4.3. *Let* X *and* Y *be Polish spaces and let* Z *be a second countable metrizable space. If* $f\colon X \to Y$ *is a class* 1*, closed map and* $g\colon X \to Z$ *of class* α, *then there is a class* 1 *map* $s\colon Y \to X$ *such that*

(i) $f(s(y)) = y$ *for all* $y \in Y$,

(ii) $g \circ s$ *is of class* α.

PROOF. As in the case of Theorem 4.1, we need to prove the result for $\alpha = 1$ only.

Case f *is continuous.* Consider the multifunction $F(y) = f^{-1}(y), y \in Y$. Apply Theorem 4.1 for $T = Y, F$ and g.

Case f *is of class* 1. Let $\tilde{X}$ be graph(f), $\tilde{f}$ the projection $\pi_Y\colon \tilde{X} \to Y$ and $\tilde{g} = g \circ \pi_X$. By a result of Jayne and Rogers [**4**, Lemma 5], the map f is closed. We now use the previous case, get a class 1 map $\tilde{s}\colon Y \to \tilde{X}$ such that $\pi_Y(\tilde{s}(y)) = y$ for all y and $\tilde{g} \circ \tilde{s}$ is of class 1. Put $s = \pi_X \circ \tilde{s}$.

THEOREM 4.4. *The previous theorm is true for an open class* 1 *map* f *also.*

PROOF. As in the previous theorem, we can assume $\alpha = 1$ and f continuous. By the theorem of Michael, there is an $X' \subseteq X$ such that $f(X') = Y$ and the restriction of f to X' is perfect (or proper). It follows that X' is Polish [**2**]. Now use the previous theorem for $f\colon X' \to Y$ and $g\colon X' \to Z$.

The idea of using projection to deduce the results for class 1 maps from that in continuous maps is due to Jayne and Rogers. However, we can use Theorem 3.2 to get the results for class 1 closed maps and extend the result of Michael for class 1 open maps.

5. Invariance of Borel pointclasses. As an application of our results, we deduce the invariance results for Borel pointclasses (obtained by Saint-Raymond, Jayne and Rogers, and Kunen and Miller [**5**]).

THEOREM 5.1. *Let X, Y be Polish spaces and let $f\colon X \to Y$ be a class 1 surjection which is either open or closed. If $E \subseteq Y$ is such that $f^{-1}(E)$ is a Borel set of multiplicative class α or of additive class α then E is of the same class as that of $f^{-1}(E)$.*

PROOF. The proof is sufficient to prove the result for multiplicative class α. So assume $E \subseteq Y$ and $f^{-1}(E)$ is of multiplicative class α. Get a class α map $g\colon X \to [0,1]$ such that $g^{-1}(0) = f^{-1}(E)$. By our results, there is a class 1 map $s\colon Y \to X$ such that $g \circ s$ is of class α and $f(s(y)) = y$ for all y. Hence, $E = (g \circ s)^{-1}(0)$ is of multiplicative class α.

An examination of the proof shows that the above theorem is also true when f is a closed map and $f^{-1}(y)$ is closed in X for each $y \in Y$.

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