

SELECTION THEOREMS AND INVARIANCE OF BOREL POINTCLASSES

SREELA BHATTACHARYA AND S. M. SRIVASTAVA

ABSTRACT. We generalize some known selection theorems and give simple proofs of results on the invariance of Borel pointclasses obtained by Saint-Raymond, Jayne and Rogers, and Kunen and Miller.

1. Introduction. In [9], Saint-Raymond proved the following selection theorem.

THEOREM 1.1. *Suppose X and Y are compact metrizable spaces and Z is a second countable metrizable space. If $f: X \rightarrow Y$ is a continuous surjection and $p: X \rightarrow Z$ is a Borel measurable function of class α , then there is a class 1 map $s: Y \rightarrow X$ such that $p \circ s$ is of class α and $f(s(y)) = y$ for all y .*

Jayne and Rogers [3, 4] have made a detailed study of this result and have extended it to the nonseparable case for continuous as well as for class 1 maps. Here we extend the result of Saint-Raymond in the more general set-up of Kuratowski and Ryll-Nardzewski [7] and Debs [1]. Our proof is simpler than those of Saint-Raymond, Jayne and Rogers.

As an application Saint-Raymond, Jayne and Rogers gave results on the complexity of preimages of Borel sets. For the sake of completeness we indicate these applications in our paper.

2. Notation and preliminaries. Throughout T will denote an arbitrary set, \mathfrak{A} a family of subsets of T , and X, Y, Z second countable metrizable spaces. A second countable, completely metrizable space is called a *Polish space*. Given $\mathfrak{A}, \mathfrak{A}_\sigma (\mathfrak{A}_\delta)$ will denote the family of unions (intersections) of a sequence of sets in \mathfrak{A} .

A *multifunction* $F: T \rightarrow X$ is a map defined on T whose values are nonempty subsets of X . For $E \subseteq X$,

$$F^{-1}(E) = \{t \in T: F(t) \cap E \neq \emptyset\}.$$

We say that F is \mathfrak{A} -*measurable* (*strongly \mathfrak{A} -measurable*) if $F^{-1}(E) \in \mathfrak{A}$ for every open (closed) set E in X . In particular, a point map $f: T \rightarrow X$ is \mathfrak{A} -measurable if $f^{-1}(E) \in \mathfrak{A}$ for every open subset E of X . A function $s: T \rightarrow X$ is called a *selector* for F if $s(t) \in F(t)$ for every $t \in T$. The set

$$\{(t, x) \in T \times X: x \in F(t)\}$$

is called the *graph* of F and is denoted by $\text{Graph}(F)$. If T is a topological space then $F: T \rightarrow X$ is called *lower semicontinuous* (*upper semicontinuous*) if $F^{-1}(A)$ is open (closed) for every open (closed) set A in X .

Received by the editors June 21, 1985.

1980 *Mathematics Subject Classification*. Primary 03E15, 54C60, 54C65.

Key words and phrases. Set-valued maps, measurable selections, Borel sets.

For notation and terminology in descriptive set theory we follow Kuratowski [6]. The set of natural numbers will be denoted by ω . Further, $\omega^{<\omega} = \bigcup_{k \in \omega} \omega^k$. We shall use e to denote the empty sequence. A map $f: X \rightarrow Y$ is called an *open (closed) map* if $f(X) = Y$ and the image of every open (closed) set in X is open (closed) in Y . For $A \subseteq X$, $\text{cl}(A)$ will denote the closure of A in X .

3. Main results. From now on, \mathcal{L} is a field of subsets of T, X and Y are Polish spaces and Z is a second countable metrizable space.

THEOREM 3.1. *If $F: T \rightarrow X$ is a closed valued, strongly \mathcal{L}_σ -measurable multifunction and $g: X \rightarrow Z$ is a class 1 map, then there is an \mathcal{L}_σ -measurable selector $s: T \rightarrow X$ for F such that $g \circ s$ is also \mathcal{L}_σ -measurable.*

PROOF. Fix a complete metric d on X compatible with its topology such that $d\text{-diameter}(X) < 1$.

Step 1. There are systems of subsets $\{T(s) : s \in \omega^{<\omega}\}$ and $\{H(s) : s \in \omega^{<\omega}\}$ of T and X respectively such that, for every $s \in \omega^k$ and $n, m \in \omega$ ($n \neq m$),

(i) $T(s) \in \mathcal{L}_\sigma, T(e) = T,$

(ii) $T(sn) \cap T(sm) = \emptyset,$

(iii) $T(s) = \bigcup_{j \in \omega} T(sj),$

(iv) $H(s)$ is closed and of d -diameter $< 2^{-k},$

(v) $d'\text{-diameter}(g(H(s))) < 1/2^k$ (where d' is a fixed metric on Z compatible with its topology),

(vii) for every $t \in T(s), F(t) \cap H(s) \neq \emptyset.$

To see that such systems exist we shall proceed inductively. Define $T(e) = T$ and $H(e) = X$. Suppose for every $s \in \bigcup_{i=0}^k \omega^i, T(s)$ and $H(s)$ have been defined satisfying (i)–(vii). Fix an $s \in \omega^k$. Fix a base W_0, W_1, \dots for Z such that $d'\text{-diameter}(W_i) < 2^{-(k+1)}$ for all i . Also, fix a base V_0, V_1, \dots for X such that, for each $i, d\text{-diameter}(V_i) < 2^{-(k+1)}$.

Let $g^{-1}(W_i) = \bigcup_{j \in \omega} E_{ij}$, where E_{ij} are closed in X . Enumerate $\{E_{ij} : i \in \omega, j \in \omega\}$ in a single sequence F_0, F_1, \dots . For $m, n \in \omega$, let

$$T'(s)(n, m) = \{t \in T(s) : F(t) \cap H(s) \cap \text{cl}(V_m) \cap F_n \neq \emptyset\}.$$

Since F is strongly \mathcal{L}_σ -measurable and $T(s) \in \mathcal{L}_\sigma, T'(s)(n, m) \in \mathcal{L}_\sigma, \forall m, n \in \omega$. Enumerate $\{T'(s)(n, m) : m, n \in \omega\}$ in a single sequence $\{T''(s)(n) : n \in \omega\}$. By [7] we get pairwise disjoint sets $T(sn), n \in \omega$, in \mathcal{L}_σ such that

$$\bigcup_{n \in \omega} T(sn) = \bigcup_{n \in \omega} T''(s)(n)$$

and

$$T(sn) \subseteq T''(s)(n) \quad \text{for all } n.$$

Put

$$H(sn) = H(s) \cap \text{cl}(V_j) \cap F_i$$

whenever $T(sn) \subseteq T'(s)(i, j)$. This completes the definition of two systems of sets with required properties.

Step 2. We now define the selector s for F . Fix $t \in T$. There is a unique $\alpha \in \omega^\omega$ such that $t \in T(\alpha|k)$ for all k . By (iv) and (v), $\bigcap_{k \in \omega} H(\alpha|k)$ is a singleton. We define $s(t)$ to be the unique point of $\bigcap_{k \in \omega} H(\alpha|k)$.

To prove that s is \mathcal{L}_σ -measurable, fix a closed set $E \subseteq X$. Let

$$E_k = \{x \in X : (\exists y \in E)(d(y, x) < 1/2^k)\}, \quad k \in \omega.$$

Also, note that for each k , the multifunction $F_k: T \rightarrow X$ defined by

$$F_k(t) = F(t) \cap H(n_0, \dots, n_{k-1}) \quad \text{if } t \in T(n_0, \dots, n_{k-1})$$

is strongly \mathcal{L}_σ -measurable. It is easy to check that

$$s^{-1}(E) = \bigcap_{k \in \omega} \{t \in T : F_k(t) \subseteq E_k\}.$$

Hence, $s^{-1}(E) \in \mathcal{L}_\delta$. Thus, s is \mathcal{L}_σ -measurable.

Now, it only remains to show that $g \circ s$ is \mathcal{L}_σ -measurable. To see this choose a point $x(n_0, \dots, n_{k-1}) \in H(n_0, \dots, n_{k-1})$ for each (n_0, \dots, n_{k-1}) . Define $f_k: T \rightarrow X$ by

$$f_k(t) = x(n_0, n_1, \dots, n_{k-1}) \quad \text{if } t \in T(n_0, \dots, n_{k-1}).$$

It is easy to see that $g \circ f_k$ is \mathcal{L}_σ -measurable. Further, by (v), $\{g \circ f_k\}_{k \in \omega}$ converges uniformly to $g \circ s$. Hence, $g \circ s$ is \mathcal{L}_σ -measurable [7]. The proof is complete.

We now extend this theorem for G_δ -valued multifunctions.

THEOREM 3.2. *Let T, X, Z, \mathcal{L} and g be as in the previous theorem. Suppose $F: T \rightarrow X$ is a strongly \mathcal{L}_σ -measurable multifunction such that*

$$\text{Graph}(F) \in (\mathcal{L} \times \mathcal{U})_{\sigma\delta}$$

where

$$\mathcal{L} \times \mathcal{U} = \{E \times U \subseteq T \times X : E \in \mathcal{L} \text{ and } U \subseteq X \text{ is open}\}.$$

Then there is an \mathcal{L}_σ -measurable selector $s: T \rightarrow X$ for F such that $g \circ s$ is also \mathcal{L}_σ -measurable.

PROOF. Set $\text{Graph}(F) = \bigcap_{k=1}^\infty G_k$, where $G_k = \bigcup_{n \in \omega} (E_{nk} \times U_{nk})$, $E_{nk} \in \mathcal{L}$ and U_{nk} open in X , for all n and k .

A slight modification of the argument in Step 1 of the previous theorem gives us a system of sets $\{T(s) : s \in \omega^{<\omega}\}$ and $\{H(s) : s \in \omega^{<\omega}\}$ in T and X respectively which satisfy conditions (i)–(vii) and

(viii) for every $k \in \omega$, $\bigcup_{s \in \omega^k} \{T(s) \times H(s)\} \subseteq G_k$.

Now we follow the arguments of Theorem 3.1 and obtain a selector s with required properties.

REMARK. The set up of Theorem 3.2 is very similar to that of the main theorem in [1].

4. Selection theorems for lower semicontinuous and upper semicontinuous multifunctions. As a corollary to Theorem 3.1, we now give several generalizations to the selection theorem of Saint-Raymond. See also [3, 4].

THEOREM 4.1. *Let X be a Polish space and let T, Z be second countable completely metrizable spaces. Suppose $g: X \rightarrow Z$ is a Borel measurable function of class α ($1 \leq \alpha < \omega_1$) and $F: T \rightarrow X$ is an upper semicontinuous (u.s.c.) closed valued*

multifunction. Then there is a class 1 selector $s: T \rightarrow X$ for F such that $g \circ s$ is of class α .

PROOF. Since every u.s.c. closed valued multifunction admits a class 1 selector, the result is true for $\alpha > \omega_0$. To prove the result for finite α , we proceed by induction.

Suppose $\alpha = 1$. Take \mathcal{L} to be the family of subsets of T which are simultaneously F_σ and G_δ and apply Theorem 3.1.

Assume the result is true for $\alpha = m$. If g is of class $(m + 1)$ then there is a sequence of class m function $g_n: X \rightarrow Z$ which converges pointwise to g . Define $h: X \rightarrow Z^\omega$ by

$$h(x) = (g_0(x), g_1(x), \dots), \quad x \in X.$$

By induction hypothesis, we get a class 1 selector $s: T \rightarrow X$ for F such that $h \circ s$ is of class m . Then $g \circ s = \lim_n g_n \circ s$, and hence is of class $(m + 1)$.

REMARK. The induction argument above is due to Saint-Raymond.

THEOREM 4.2. *The previous theorem is also true when F is lower semicontinuous (l.s.c.).*

PROOF. We need to prove the result for $\alpha = 1$ only. By a result of Michael [8] there is a compact valued, u.s.c. multifunction $H: T \rightarrow X$ such that $H(t) \subseteq F(t)$ for all t . We use Theorem 4.1 for H and g .

THEOREM 4.3. *Let X and Y be Polish spaces and let Z be a second countable metrizable space. If $f: X \rightarrow Y$ is a class 1, closed map and $g: X \rightarrow Z$ of class α , then there is a class 1 map $s: Y \rightarrow X$ such that*

- (i) $f(s(y)) = y$ for all $y \in Y$,
- (ii) $g \circ s$ is of class α .

PROOF. As in the case of Theorem 4.1, we need to prove the result for $\alpha = 1$ only.

Case f is continuous. Consider the multifunction $F(y) = f^{-1}(y), y \in Y$. Apply Theorem 4.1 for $T = Y, F$ and g .

Case f is of class 1. Let \tilde{X} be $\text{graph}(f)$, \tilde{f} the projection $\pi_Y: \tilde{X} \rightarrow Y$ and $\tilde{g} = g \circ \pi_X$. By a result of Jayne and Rogers [4, Lemma 5], the map f is closed. We now use the previous case, get a class 1 map $\tilde{s}: Y \rightarrow \tilde{X}$ such that $\pi_Y(\tilde{s}(y)) = y$ for all y and $\tilde{g} \circ \tilde{s}$ is of class 1. Put $s = \pi_X \circ \tilde{s}$.

THEOREM 4.4. *The previous theorem is true for an open class 1 map f also.*

PROOF. As in the previous theorem, we can assume $\alpha = 1$ and f continuous. By the theorem of Michael, there is an $X' \subseteq X$ such that $f(X') = Y$ and the restriction of f to X' is perfect (or proper). It follows that X' is Polish [2]. Now use the previous theorem for $f: X' \rightarrow Y$ and $g: X' \rightarrow Z$.

The idea of using projection to deduce the results for class 1 maps from that in continuous maps is due to Jayne and Rogers. However, we can use Theorem 3.2 to get the results for class 1 closed maps and extend the result of Michael for class 1 open maps.

5. Invariance of Borel pointclasses. As an application of our results, we deduce the invariance results for Borel pointclasses (obtained by Saint-Raymond, Jayne and Rogers, and Kunen and Miller [5]).

THEOREM 5.1. *Let X, Y be Polish spaces and let $f: X \rightarrow Y$ be a class 1 surjection which is either open or closed. If $E \subseteq Y$ is such that $f^{-1}(E)$ is a Borel set of multiplicative class α or of additive class α then E is of the same class as that of $f^{-1}(E)$.*

PROOF. The proof is sufficient to prove the result for multiplicative class α . So assume $E \subseteq Y$ and $f^{-1}(E)$ is of multiplicative class α . Get a class α map $g: X \rightarrow [0, 1]$ such that $g^{-1}(0) = f^{-1}(E)$. By our results, there is a class 1 map $s: Y \rightarrow X$ such that $g \circ s$ is of class α and $f(s(y)) = y$ for all y . Hence, $E = (g \circ s)^{-1}(0)$ is of multiplicative class α .

An examination of the proof shows that the above theorem is also true when f is a closed map and $f^{-1}(y)$ is closed in X for each $y \in Y$.

REFERENCES

1. G. Debs, *Selection d'une multi-application a valeurs \mathcal{G}_δ* , Acad Roy. Belg. Bull. Cl. Sci **65** (1979), 211–216.
2. R. Engelking, *General topology*, PWN, Warsaw, 1977.
3. J. E. Jayne and C. A. Rogers, *Invariance of Borel classes in metric spaces*, Math. Ann. **263** (1983), 323–341.
4. ———, *The invariance of absolute Borel classes, Convex analysis and applications*, Research Notes in Math., No. 57, Pitman, London, 1982, pp. 118–151.
5. K. Kunen and A. N. Miller, *Borel and projective sets from the point of view of compact sets*, Math. Proc. Cambridge Philos. Soc. **94** (1983), 399–409.
6. K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
7. K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **13** (1961), 397–403.
8. E. Michael, *A theorem on semi-continuous set-valued functions*, Duke Math. J. **26** (1959), 647–651.
9. J. Saint-Raymond, *Fonctions Boreliennes sur un quotient*, Bull. Soc. Math. France (2) **100** (1976), 141–147.

STATISTICS AND MATHEMATICS DIVISION, INDIAN STATISTICAL INSTITUTE, 203 B.T. ROAD, CALCUTTA 700 035, INDIA