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SPURIOUS BROWNIAN MOTIONS[1]

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ABSTRACT. Spurious Brownian motions are characterized in $\mathbf{R}^d$, $d \geqslant 2$.

Let $W = \{W(t) = (W_1(t), \dots, W_d(t)),\ t \geqslant 0\}$ be an $\mathbf{R}^d$-valued mean-zero Gaussian process such that all projections $Y_\lambda(t) = \sum_{j=1}^d \lambda_j W_j(t)$ behave as if W were standard Brownian motion in $\mathbf{R}^d$, i.e., $EY_\lambda(s)Y_\lambda(t) = \min(s,t)\sum_{j=1}^d \lambda_j^2$. Following Hardin [**1**], we call W a spurious Brownian motion if it is not a standard Brownian motion in $\mathbf{R}^d$. He showed by an example that such a process exists in $\mathbf{R}^2$.

Let $S(s,t) = (\sigma_{jk}(s,t))$ denote the covariance matrix function of W: $\sigma_{jk}(s,t) = EW_j(s)W_k(t)$, $j,k = 1,\dots,d$. Necessarily, $\sigma_{jj}(s,t) = \min(s,t)$ for each $j = 1,\dots,d$. Hence the identity $EY_\lambda(s)Y_\lambda(t) = \sum_{j,k=1}^d \lambda_j\lambda_k\sigma_{jk}(s,t) \equiv \min(s,t)\sum_{j=1}^d \lambda_j^2$ is equivalent to the identity

$$\sum_{j,k=1;\, j\neq k}^{d} \lambda_j\lambda_k\sigma_{jk}(s,t) \equiv 0.$$

This holds for any vector $\lambda = (\lambda_1,\dots,\lambda_d)$ if and only if $\sigma_{jk}(s,t) \equiv -\sigma_{kj}(s,t)$ for all $j,k = 1,\dots,d$; $j \neq k$. Thus W is a spurious Brownian motion if and only if $S(s,t)$ is skew-symmetric and not diagonal.

Hardin's example in $\mathbf{R}^2$ is an example for such a covariance matrix with $\sigma_{12}(s,t) = 3^{-1}\{\min(2s,t) - \min(s,2t)\}$. However, for any choice of the functions $\sigma_{jk}(s,t)$, $k = j+1,\dots,d$; $j = 1,\dots,d$, such that $|\sigma_{jk}(s,t)| \leqslant (st)^{1/2}$, $\sigma_{jk}(s,t) + \sigma_{jk}(t,s) \equiv 0$, and that $\sigma_{jk}(s,t) \not\equiv 0$ for at least one pair (j,k), there is a spurious Brownian motion in $\mathbf{R}^d$, provided that for any integer $m \geqslant 2$, any $t_1,\dots,t_m \geqslant 0$, and any real numbers λ_{il}, $i = 1,\dots,d$; $l = 1,\dots,m$, we have

$$\sum_{i,l=1}^{m}\left\{\min(t_i,t_l)\sum_{j=1}^{d}\lambda_{ji}\lambda_{jl} + \sum_{j=1}^{d-1}\sum_{k=j+1}^{d}\sigma_{jk}(t_i,t_l)\left(\lambda_{ji}\lambda_{kl} - \lambda_{jl}\lambda_{ki}\right)\right\} \geqslant 0.$$

These conditions are necessary and sufficient for the existence of a mean-zero stochastic process $X = \{X(t) = (X_1(t),\dots,X_d(t)), t \geqslant 0\}$ that has covariance matrix function defined as the skew-symmetric matrix corresponding to the functions $\sigma_{jk}(s,t)$, $k = j+1,\dots,d$; $j = 1,\dots,d$, and having diagonal $\sigma_{jj}(s,t) = \min(s,t)$,

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$j = 1, \ldots, d$. The projections of X already behave as if X were a standard Brownian motion in $\mathbf{R}^d$, though X is not necessarily Gaussian. If X is Gaussian, and such an X exists, then it is necessarily a spurious Brownian motion in $\mathbf{R}^d$.

REFERENCES

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